Chapter 4: Probability

Section 4.1: Empirical Probability

One story about how probability theory was developed is that a gambler wanted to know when to bet more and when to bet less. He talked to a couple of friends of his that happened to be mathematicians. Their names were Pierre de Fermat and Blaise Pascal. Since then many other mathematicians have worked to develop probability theory.

Understanding probabilities are important in life. Examples of mundane questions that probability can answer for you are if you need to carry an umbrella or wear a heavy coat on a given day. More important questions that probability can help with are your chances that the car you are buying will need more maintenance, your chances of passing a class, your chances of winning the lottery, your chances of being in a car accident, and the chances that the U.S. will be attacked by terrorists. Most people do not have a very good understanding of probability, so they worry about being attacked by a terrorist but not about being in a car accident. The probability of being in a terrorist attack is much smaller than the probability of being in a car accident, thus it actually would make more sense to worry about driving. Also, the chance of you winning the lottery is very small, yet many people will spend the money on lottery tickets. Yet, if instead they saved the money that they spend on the lottery, they would have more money. In general, events that have a low probability (under 5%) are unlikely to occur. Whereas if an event has a high probability of happening (over 80%), then there is a good chance that the event will happen. This chapter will present some of the theory that you need to help make a determination of whether an event is likely to happen or not.

First you need some definitions.
Experiment: an activity that has specific results that can occur, but it is unknown which results will occur.
Outcomes: the results of an experiment
Event: a set of certain outcomes of an experiment that you want to have happen
Sample Space: collection of all possible outcomes of the experiment. Usually denoted as SS.
Event space: the set of outcomes that make up an event. The symbol is usually a capital letter.

Start with an experiment. Suppose that the experiment is rolling a die. The sample space is {1, 2, 3, 4, 5, 6}. The event that you want is to get a 6, and the event space is {6}. To do this, roll a die 10 times. When you do that, you get a 6 two times. Based on this experiment, the probability of getting a 6 is 2 out of 10 or 1/5. To get more accuracy, repeat the experiment more times. It is easiest to put this in a table, where n represents the number of times the experiment is repeated. When you put the number of 6s found over the number of times you repeat the experiment, this is the relative frequency.

Table #4.1.1: Trials for Die Experiment

n	Number of 6s	Relative Frequency
10	2	0.2
50	6	0.12
100	18	0.18
500	81	0.162
1000	163	0.163

Notice that as n increased, the relative frequency seems to approach a number. It looks like it is approaching 0.163. You can say that the probability of getting a 6 is approximately 0.163. If you want more accuracy, then increase n even more.

These probabilities are called **experimental probabilities** since they are found by actually doing the experiment. They come about from the relative frequencies and give an approximation of the true probability. The approximate probability of an event A, $P(A)$, is

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Experimental Probabilities

$$P(A) = \frac{\text{number of times } A \text{ occurs}}{\text{number of times the experiment was repeated}}$$

For the event of getting a 6, the probability would be $\dfrac{163}{1000} = 0.163$.

You must do experimental probabilities whenever it is not possible to calculate probabilities using other means. An example is if you want to find the probability that a family has 5 children, you would have to actually look at many families, and count how many have 5 children. Then you could calculate the probability. Another example is if you want to figure out if a die is fair. You would have to roll the die many times and count how often each side comes up. Make sure you repeat an experiment many times, because otherwise you will not be able to estimate the true probability. This is due to the law of large numbers.

Law of large numbers: as n increases, the relative frequency tends towards the actual probability value.

Note: probability, relative frequency, percentage, and proportion are all different words for the same concept. Also, probabilities can be given as percentages, decimals, or fractions.

Section 4.1: Homework

1.) Table #4.1.2 contains the number of M&M's of each color that were found in a case (Madison, 2013).

Table #4.1.2: M&M Distribution

Blue	Brown	Green	Orange	Red	Yellow	Total
481	371	483	544	372	369	2620

Find the probability of choosing each color based on this experiment.

2.) Eyeglassomatic manufactures eyeglasses for different retailers. They test to see how many defective lenses they made the time period of January 1 to March 31. Table #4.1.3 gives the defect and the number of defects.

Table #4.1.3: Number of Defective Lenses

Defect type	Number of defects
Scratch	5865
Right shaped – small	4613
Flaked	1992
Wrong axis	1838
Chamfer wrong	1596
Crazing, cracks	1546
Wrong shape	1485
Wrong PD	1398
Spots and bubbles	1371
Wrong height	1130
Right shape – big	1105
Lost in lab	976
Spots/bubble – intern	976

Find the probability of each defect type based on this data.

3.) In Australia in 1995, of the 2907 indigenous people in prison 17 of them died. In that same year, of the 14501 non-indigenous people in prison 42 of them died ("Aboriginal deaths in," 2013). Find the probability that an indigenous person dies in prison and the probability that a non-indigenous person dies in prison. Compare these numbers and discuss what the numbers may mean.

4.) A project conducted by the Australian Federal Office of Road Safety asked people many questions about their cars. One question was the reason that a person chooses a given car, and that data is in table #4.1.4 ("Car preferences," 2013).

Table #4.1.4: Reason for Choosing a Car

Safety	Reliability	Cost	Performance	Comfort	Looks
84	62	46	34	47	27

Find the probability a person chooses a car for each of the given reasons.

Section 4.2: Theoretical Probability

It is not always feasible to conduct an experiment over and over again, so it would be better to be able to find the probabilities without conducting the experiment. These probabilities are called **Theoretical Probabilities.**

To be able to do theoretical probabilities, there is an assumption that you need to consider. It is that all of the outcomes in the sample space need to be **equally likely outcomes**. This means that every outcome of the experiment needs to have the same chance of happening.

Example #4.2.1: Equally Likely Outcomes
Which of the following experiments have equally likely outcomes?
a.) Rolling a fair die.

Solution:
Since the die is fair, every side of the die has the same chance of coming up. The outcomes are the different sides, so each outcome is equally likely

b.) Flip a coin that is weighted so one side comes up more often than the other.

Solution:
Since the coin is weighted, one side is more likely to come up than the other side. The outcomes are the different sides, so each outcome is not equally likely

c.) Pull a ball out of a can containing 6 red balls and 8 green balls. All balls are the same size.

Solution:
Since each ball is the same size, then each ball has the same chance of being chosen. The outcomes of this experiment are the individual balls, so each outcome is equally likely. Don't assume that because the chances of pulling a red ball are less than pulling a green ball that the outcomes are not equally likely. The outcomes are the individual balls and they are equally likely.

d.) Picking a card from a deck.

Solution:
If you assume that the deck is fair, then each card has the same chance of being chosen. Thus the outcomes are equally likely outcomes. You do have to make this assumption. For many of the experiments you will do, you do have to make this kind of assumption.

e.) Rolling a die to see if it is fair.

Solution:

In this case you are not sure the die is fair. The only way to determine if it is fair is to actually conduct the experiment, since you don't know if the outcomes are equally likely. If the experimental probabilities are fairly close to the theoretical probabilities, then the die is fair.

If the outcomes are not equally likely, then you must do experimental probabilities. If the outcomes are equally likely, then you can do theoretical probabilities.

Theoretical Probabilities: If the outcomes of an experiment are equally likely, then the probability of event A happening is

$$P(A) = \frac{\text{\# of outcomes in event space}}{\text{\# of outcomes in sample space}}$$

Example #4.2.2: Calculating Theoretical Probabilities

Suppose you conduct an experiment where you flip a fair coin twice

a.) What is the sample space?

Solution:

There are several different sample spaces you can do. One is
SS={0, 1, 2} where you are counting the number of heads. However, the outcomes are not equally likely since you can get one head by getting a head on the first flip and a tail on the second or a tail on the first flip and a head on the second. There are 2 ways to get that outcome and only one way to get the other outcomes. Instead it might be better to give the sample space as listing what can happen on each flip. Let H = head and T = tail, and list which can happen on each flip.
SS={HH, HT, TH, TT}

b.) What is the probability of getting exactly one head?

Solution:

Let A = getting exactly one head. The event space is A = {HT, TH}. So
$$P(A) = \frac{2}{4} \text{ or } \frac{1}{2}.$$
It may not be advantageous to reduce the fractions to lowest terms, since it is easier to compare fractions if they have the same denominator.

c.) What is the probability of getting at least one head?

Solution:

Let B = getting at least one head. At least one head means get one or more. The event space is B = {HT, TH, HH} and

$$P(B) = \frac{3}{4}$$

Since P(B) is greater than the P(A), then event B is more likely to happen than event A.

d.) What is the probability of getting a head and a tail?

Solution:

Let C = getting a head and a tail = {HT, TH} and

$$P(C) = \frac{2}{4}$$

This is the same event space as event A, but it is a different event. Sometimes two different events can give the same event space.

e.) What is the probability of getting a head or a tail?

Solution:

Let D = getting a head or a tail. Since or means one or the other or both and it doesn't specify the number of heads or tails, then D = {HH, HT, TH, TT} and

$$P(D) = \frac{4}{4} = 1$$

f.) What is the probability of getting a foot?

Solution:

Let E = getting a foot. Since you can't get a foot, E = {} or the empty set and

$$P(E) = \frac{0}{4} = 0$$

g.) What is the probability of each outcome? What is the sum of these probabilities?

Solution:

$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$. If you add all of these

probabilities together you get 1.

This example had some results in it that are important concepts. They are summarized below:

Probability Properties
1. $0 \le P(\text{event}) \le 1$
2. If the P(event) = 1, then it will happen and is called the certain event
3. If the P(event) = 0, then it cannot happen and is called the impossible event
4. $\sum P(\text{outcome}) = 1$

Example #4.2.3: Calculating Theoretical Probabilities
Suppose you conduct an experiment where you pull a card from a standard deck.
a.) What is the sample space?

Solution:
SS = {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS, 2C, 3C, 4C, 5C, 6C, 7C, 8C, 9C, 10C, JC, QC, KC, AC, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD, AD, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AH}

b.) What is the probability of getting a Spade?

Solution:
Let A = getting a spade = {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS} so
$$P(A) = \frac{13}{52}$$

c.) What is the probability of getting a Jack?

Solution:
Let B = getting a Jack = {JS, JC, JH, JD} so
$$P(B) = \frac{4}{52}$$

d.) What is the probability of getting an Ace?

Solution:
Let C = getting an Ace = {AS, AC, AH, AD} so
$$P(C) = \frac{4}{52}$$

e.) What is the probability of not getting an Ace?

Solution:
Let D = not getting an Ace = {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, 2C, 3C, 4C, 5C, 6C, 7C, 8C, 9C, 10C, JC, QC, KC, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH} so

$$P(D) = \frac{48}{52}$$

Notice, $P(D) + P(C) = \frac{48}{52} + \frac{4}{52} = 1$, so you could have found the probability of D by doing 1 minus the probability of C

$$P(D) = 1 - P(C) = 1 - \frac{4}{52} = \frac{48}{52}$$

f.) What is the probability of getting a Spade and an Ace?

Solution:
Let E = getting a Spade and an Ace = {AS} so

$$P(E) = \frac{1}{52}$$

g.) What is the probability of getting a Spade or an Ace?

Solution:
Let F = getting a Spade and an Ace ={2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS, AC, AD, AH} so

$$P(F) = \frac{16}{52}$$

h.) What is the probability of getting a Jack and an Ace?

Solution:
Let G = getting a Jack and an Ace = { } since you can't do that with one card. So

$$P(G) = 0$$

i.) What is the probability of getting a Jack or an Ace?

Solution:
Let H = getting a Jack or an Ace = {JS, JC, JD, JH, AS, AC, AD, AH} so

$$P(H) = \frac{8}{52}$$

Example #4.2.4: Calculating Theoretical Probabilities

Suppose you have an iPod Shuffle with the following songs on it: 5 Rolling Stones songs, 7 Beatles songs, 9 Bob Dylan songs, 4 Faith Hill songs, 2 Taylor Swift songs, 7 U2 songs, 4 Mariah Carey songs, 7 Bob Marley songs, 6 Bunny Wailer songs, 7 Elton John songs, 5 Led Zeppelin songs, and 4 Dave Mathews Band songs. The different genre that you have are rock from the 60s which includes Rolling Stones, Beatles, and Bob Dylan; country includes Faith Hill and Taylor Swift; rock of the 90s includes U2 and Mariah Carey; Reggae includes Bob Marley and Bunny Wailer; rock of the 70s includes Elton John and Led Zeppelin; and bluegrasss/rock includes Dave Mathews Band.

The way an iPod Shuffle works, is it randomly picks the next song so you have no idea what the next song will be. Now you would like to calculate the probability that you will hear the type of music or the artist that you are interested in. The sample set is too difficult to write out, but you can figure it from looking at the number in each set and the total number. The total number of songs you have is 67.

a.) What is the probability that you will hear a Faith Hill song?

Solution:
There are 4 Faith Hill songs out of the 67 songs, so
$$P(\text{Faith Hill song}) = \frac{4}{67}$$

b.) What is the probability that you will hear a Bunny Wailer song?

Solution:
There are 6 Bunny Wailer songs, so
$$P(\text{Bunny Wailer}) = \frac{6}{67}$$

c.) What is the probability that you will hear a song from the 60s?

Solution:
There are 5, 7, and 9 songs that are classified as rock from the 60s, which is 21 total, so
$$P(\text{rock from the 60s}) = \frac{21}{67}$$

d.) What is the probability that you will hear a Reggae song?

Solution:
There are 6 and 7 songs that are classified as Reggae, which is 13 total, so

$$P(\text{Reggae}) = \frac{13}{67}$$

e.) What is the probability that you will hear a song from the 90s or a bluegrass/rock song?

Solution:
There are 7 and 4 songs that are songs from the 90s and 4 songs that are bluegrass/rock, for a total of 15, so

$$P(\text{rock from the 90s or bluegrass/rock}) = \frac{15}{67}$$

f.) What is the probability that you will hear an Elton John or a Taylor Swift song?

Solution:
There are 7 Elton John songs and 2 Taylor Swift songs, for a total of 9, so

$$P(\text{Elton John or Taylor Swift song}) = \frac{9}{67}$$

g.) What is the probability that you will hear a country song or a U2 song?

Solution:
There are 6 country songs and 7 U2 songs, for a total of 13, so

$$P(\text{country or U2 song}) = \frac{13}{67}$$

Of course you can do any other combinations you would like.

Notice in example #4.2.3 part e, it was mentioned that the probability of event D plus the probability of event C was 1. This is because these two events have no outcomes in common, and together they make up the entire sample space. Events that have this property are called **complementary events**.

If two events are **complementary events** then to find the probability of one just subtract the probability of the other from one. Notation used for complement of A is not A or A^C

$$P(A) + P(A^C) = 1, \text{ or } P(A) = 1 - P(A^C).$$

Example #4.2.5: Complementary Events

a.) Suppose you know that the probability of it raining today is 0.45. What is the probability of it not raining?

Solution:

Since not raining is the complement of raining, then
$$P(\text{not raining}) = 1 - P(\text{raining}) = 1 - 0.45 = 0.55$$

b.) Suppose you know the probability of not getting the flu is 0.24. What is the probability of getting the flu?

Solution:

Since getting the flu is the complement of not getting the flu, then
$$P(\text{getting the flu}) = 1 - P(\text{not getting the flu}) = 1 - 0.24 = 0.76$$

c.) In an experiment of picking a card from a deck, what is the probability of not getting a card that is a Queen?

Solution:

You could do this problem by listing all the ways to not get a queen, but that set is fairly large. One advantage of the complement is that it reduces the workload. You use the complement in many situations to make the work shorter and easier. In this case it is easier to list all the ways to get a Queen, find the probability of the Queen, and then subtract from one. Queen = {QS, QC, QD, QH} so
$$P(\text{Queen}) = \frac{4}{52} \text{ and}$$
$$P(\text{not Queen}) = 1 - P(\text{Queen}) = 1 - \frac{4}{52} = \frac{48}{52}$$

The complement is useful when you are trying to find the probability of an event that involves the words at least or an event that involves the words at most. As an example of an at least event is suppose you want to find the probability of making at least $50,000 when you graduate from college. That means you want the probability of your salary being greater than or equal to $50,000. An example of an at most event is suppose you want to find the probability of rolling a die and getting at most a 4. That means that you want to get less than or equal to a 4 on the die. The reason to use the complement is that sometimes it is easier to find the probability of the complement and then subtract from 1. Example #4.2.6 demonstrates how to do this.

Example #4.2.6: Using the Complement to Find Probabilities
   a.) In an experiment of rolling a fair die one time, find the probability of rolling at
       most a 4 on the die.

   Solution:
   The sample space for this experiment is {1, 2, 3, 4, 5, 6}. You want the event of
   getting at most a 4, which is the same as thinking of getting 4 or less. The event
   space is {1, 2, 3, 4}. The probability is

   $$P(\text{at most } 4) = \frac{4}{6}$$

   Or you could have used the complement. The complement of rolling at most a 4
   would be rolling number bigger than 4. The event space for the complement is

   {5, 6}. The probability of the complement is $\frac{2}{6}$. The probability of at most 4

   would be

   $$P(\text{at most } 4) = 1 - P(\text{more than } 4) = 1 - \frac{2}{6} = \frac{4}{6}$$

   Notice you have the same answer, but the event space was easier to write out. On
   this example it probability wasn't that useful, but in the future there will be events
   where it is much easier to use the complement.

   b.) In an experiment of pulling a card from a fair deck, find the probability of pulling
       at least a 5 (ace is a high card in this example).

   Solution:
   The sample space for this experiment is
       SS = {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS, 2C, 3C, 4C,
       5C, 6C, 7C, 8C, 9C, 10C, JC, QC, KC, AC, 2D, 3D, 4D, 5D, 6D, 7D, 8D,
       9D, 10D, JD, QD, KD, AD, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH,
       QH, KH, AH}
   Pulling a card that is at least a 5 would involve listing all of the cards that are a 5
   or more. It would be much easier to list the outcomes that make up the
   complement. The complement of at least a 5 is less than a 5. That would be the
   event of 4 or less. The event space for the complement would be {2S, 3S, 4S, 2C,
   3C, 4C, 2D, 3D, 4D, 2H, 3H, 4H}. The probability of the complement would be

   $\frac{12}{52}$. The probability of at least a 5 would be

   $$P(\text{at least a } 5) = 1 - P(4 \text{ or less}) = 1 - \frac{12}{52} = \frac{40}{52}$$

Another concept was show in example #4.2.3 parts g and i. The problems were looking
for the probability of one event or another. In part g, it was looking for the probability of

getting a Spade or an Ace. That was equal to $\frac{16}{52}$. In part i, it was looking for the

probability of getting a Jack or an Ace. That was equal to $\dfrac{8}{52}$. If you look back at the parts b, c, and d, you might notice the following result:

$$P(\text{Jack}) + P(\text{Ace}) = P(\text{Jack or Ace}) \text{ but } P(\text{Spade}) + P(\text{Ace}) \neq P(\text{Spade or Ace}).$$

Why does adding two individual probabilities together work in one situation to give the probability of one or another event and not give the correct probability in the other?

The reason this is true in the case of the Jack and the Ace is that these two events cannot happen together. There is no overlap between the two events, and in fact the $P(\text{Jack and Ace}) = 0$. However, in the case of the Spade and Ace, they can happen together. There is overlap, mainly the ace of spades. The $P(\text{Spade and Ace}) \neq 0$.

When two events cannot happen at the same time, they are called **mutually exclusive**. In the above situation, the events Jack and Ace are mutually exclusive, while the events Spade and Ace are not mutually exclusive.

Addition Rules:
If two events A and B are mutually exclusive, then
$$P(A \text{ or } B) = P(A) + P(B) \text{ and } P(A \text{ and } B) = 0$$

If two events A and B are not mutually exclusive, then
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example #4.2.7: Using Addition Rules
Suppose your experiment is to roll two fair dice.
a.) What is the sample space?

Solution:
As with the other examples you need to come up with a sample space that has equally likely outcomes. One sample space is to list the sums possible on each roll. That sample space would look like: SS = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. However, there are more ways to get a sum of 7 then there are to get a sum of 2, so these outcomes are not equally likely. Another thought is to list the possibilities on each roll. As an example you could roll the dice and on the first die you could get a 1. The other die could be any number between 1 and 6, but say it is a 1 also. Then this outcome would look like (1,1). Similarly, you could get (1, 2), (1, 3), (1,4), (1, 5), or (1, 6). Also, you could get a 2, 3, 4, 5, or 6 on the first die instead. Putting this all together, you get the sample space:

$$SS = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$

Notice that a (2,3) is different from a (3,2), since the order that you roll the die is important and you can tell the difference between these two outcomes. You don't need any of the doubles twice, since these are not distinguishable from each other in either order.

This will always be the sample space for rolling two dice.

b.) What is the probability of getting a sum of 5?

Solution:

Let A = getting a sum of 5 = {(4,1), (3,2), (2,3), (1,4)} so

$$P(A) = \frac{4}{36}$$

c.) What is the probability of getting the first die a 2?

Solution:

Let B = getting first die a 2 = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)} so

$$P(B) = \frac{6}{36}$$

d.) What is the probability of getting a sum of 7?

Solution:

Let C = getting a sum of 7 = {(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)} so

$$P(C) = \frac{6}{36}$$

e.) What is the probability of getting a sum of 5 and the first die a 2?

Solution:

This is events A and B which contains the outcome {(2,3)} so

$$P(A \text{ and } B) = \frac{1}{36}$$

f.) What is the probability of getting a sum of 5 or the first die a 2?

Solution:

Notice from part e, that these two events are not mutually exclusive, so

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$= \frac{4}{36} + \frac{6}{36} - \frac{1}{36}$$

$$= \frac{9}{36}$$

g.) What is the probability of getting a sum of 5 and sum of 7?

Solution:

These are the events A and C, which have no outcomes in common. Thus A and C = { } so

$$P(A \text{ and } C) = 0$$

h.) What is the probability of getting a sum of 5 or sum of 7?

Solution:

From part g, these two events are mutually exclusive, so

$$P(A \text{ or } C) = P(A) + P(C)$$

$$= \frac{4}{36} + \frac{6}{36}$$

$$= \frac{10}{36}$$

Odds

Many people like to talk about the odds of something happening or not happening. Mathematicians, statisticians, and scientists prefer to deal with probabilities since odds are difficult to work with, but gamblers prefer to work in odds for figuring out how much they are paid if they win.

The **actual odds against** event A occurring are the ratio $P(A^C)/P(A)$, usually expressed in the form *a:b* or *a* to *b*, where *a* and *b* are integers with no common factors.

The **actual odds in favor** event A occurring are the ratio $P(A)/P(A^C)$, which is the reciprocal of the odds against. If the odds against event A are *a:b*, then the odds in favor event A are *b:a*.

The **payoff odds** against event A occurring are the ratio of the net profit (if you win) to the amount bet.

payoff odds against event A = (net profit) : (amount bet)

Example #4.2.8: Odds Against and Payoff Odds
In the game of Craps, if a shooter has a come-out roll of a 7 or an 11, it is called a natural and the pass line wins. The payoff odds are given by a casino as 1:1.

a.) Find the probability of a natural.

Solution:
A natural is a 7 or 11. The sample space is
$$SS = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$$
The event space is $\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (5,6), (6,5)\}$
So $P(7 \text{ or } 11) = \dfrac{8}{36}$

b.) Find the actual odds for a natural.

Solution:
$$\text{odd for a natural} = \frac{P(7 \text{ or } 11)}{P(\textit{not } 7 \text{ or } 11)}$$
$$= \frac{8/36}{1 - 8/36}$$
$$= \frac{8/36}{28/36}$$
$$= \frac{8}{28}$$
$$= \frac{2}{7}$$

c.) Find the actual odds against a natural.

Solution:
$$\text{odds against a natural} = \frac{P(\textit{not } 7 \text{ or } 11)}{P(7 \text{ or } 11)} = \frac{28}{8} = \frac{7}{2} = \frac{3.5}{1}$$

d.) If the casino pays 1:1, how much profit does the casino make on a $10 bet?

Solution:
The actual odds are 3.5 to 1 while the payoff odds are 1 to 1. The casino pays you $10 for your $10 bet. If the casino paid you the actual odds, they

would pay $3.50 on every $1 bet, and on $10, they pay $3.5*\$10 = \$35$.
Their profit is $\$35 - \$10 = \$25$.

Section 4.2: Homework

1.)   Table #4.2.1 contains the number of M&M's of each color that were found in a case (Madison, 2013).

Table #4.2.1: M&M Distribution

Blue	Brown	Green	Orange	Red	Yellow	Total
481	371	483	544	372	369	2620

a.) Find the probability of choosing a green or red M&M.
b.) Find the probability of choosing a blue, red, or yellow M&M.
c.) Find the probability of not choosing a brown M&M.
d.) Find the probability of not choosing a green M&M.

2.)   Eyeglassomatic manufactures eyeglasses for different retailers. They test to see how many defective lenses they made in a time period. Table #4.2.2 gives the defect and the number of defects.

Table #4.2.2: Number of Defective Lenses

Defect type	Number of defects
Scratch	5865
Right shaped – small	4613
Flaked	1992
Wrong axis	1838
Chamfer wrong	1596
Crazing, cracks	1546
Wrong shape	1485
Wrong PD	1398
Spots and bubbles	1371
Wrong height	1130
Right shape – big	1105
Lost in lab	976
Spots/bubble – intern	976

a.) Find the probability of picking a lens that is scratched or flaked.
b.) Find the probability of picking a lens that is the wrong PD or was lost in lab.
c.) Find the probability of picking a lens that is not scratched.
d.) Find the probability of picking a lens that is not the wrong shape.

3.)   An experiment is to flip a fair coin three times.
        a.) State the sample space.
        b.) Find the probability of getting exactly two heads. Make sure you state the
            event space.
        c.) Find the probability of getting at least two heads. Make sure you state the
            event space.
        d.) Find the probability of getting an odd number of heads. Make sure you state
            the event space.
        e.) Find the probability of getting all heads or all tails. Make sure you state the
            event space.
        f.) Find the probability of getting exactly two heads or exactly two tails.
        g.) Find the probability of not getting an odd number of heads.

4.)   An experiment is rolling a fair die and then flipping a fair coin.
        a.) State the sample space.
        b.) Find the probability of getting a head. Make sure you state the event space.
        c.) Find the probability of getting a 6. Make sure you state the event space.
        d.) Find the probability of getting a 6 or a head.
        e.) Find the probability of getting a 3 and a tail.

5.)   An experiment is rolling two fair dice.
        a.) State the sample space.
        b.) Find the probability of getting a sum of 3. Make sure you state the event
            space.
        c.) Find the probability of getting the first die is a 4. Make sure you state the
            event space.
        d.) Find the probability of getting a sum of 8. Make sure you state the event
            space.
        e.) Find the probability of getting a sum of 3 or sum of 8.
        f.) Find the probability of getting a sum of 3 or the first die is a 4.
        g.) Find the probability of getting a sum of 8 or the first die is a 4.
        h.) Find the probability of not getting a sum of 8.

6.)   An experiment is pulling one card from a fair deck.
        a.) State the sample space.
        b.) Find the probability of getting a Ten. Make sure you state the event space.
        c.) Find the probability of getting a Diamond. Make sure you state the event
            space.
        d.) Find the probability of getting a Club. Make sure you state the event space.
        e.) Find the probability of getting a Diamond or a Club.
        f.) Find the probability of getting a Ten or a Diamond.

7.) An experiment is pulling a ball from an urn that contains 3 blue balls and 5 red balls.
  a.) Find the probability of getting a red ball.
  b.) Find the probability of getting a blue ball.
  c.) Find the odds for getting a red ball.
  d.) Find the odds for getting a blue ball.

8.) In the game of roulette, there is a wheel with spaces marked 0 through 36 and a space marked 00.
  a.) Find the probability of winning if you pick the number 7 and it comes up on the wheel.
  b.) Find the odds against winning if you pick the number 7.
  c.) The casino will pay you $20 for every dollar you bet if your number comes up. How much profit is the casino making on the bet?

Section 4.3: Conditional Probability

Suppose you want to figure out if you should buy a new car. When you first go and look, you find two cars that you like the most. In your mind they are equal, and so each has a 50% chance that you will pick it. Then you start to look at the reviews of the cars and realize that the first car has had 40% of them needing to be repaired in the first year, while the second car only has 10% of the cars needing to be repaired in the first year. You could use this information to help you decide which car you want to actually purchase. Both cars no longer have a 50% chance of being the car you choose. You could actually calculate the probability you will buy each car, which is a conditional probability. You probably wouldn't do this, but it gives you an example of what a conditional probability is.

Conditional probabilities are probabilities calculated after information is given. This is where you want to find the probability of event A happening after you know that event B has happened. If you know that B has happened, then you don't need to consider the rest of the sample space. You only need the outcomes that make up event B. Event B becomes the new sample space, which is called the **restricted sample space, R**. If you always write a restricted sample space when doing conditional probabilities and use this as your sample space, you will have no trouble with conditional probabilities. The notation for conditional probabilities is $P(A, \text{given } B) = P(A|B)$. The event following the vertical line is always the restricted sample space.

Example #4.3.1: Conditional Probabilities
    a.) Suppose you roll two dice. What is the probability of getting a sum of 5, given that the first die is a 2?

        Solution:
            Since you know that the first die is a 2, then this is your restricted sample space, so
                R = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)}
            Out of this restricted sample space, the way to get a sum of 5 is {(2,3)}. Thus

$$P(\text{sum of 5}|\text{the first die is a 2}) = \frac{1}{6}$$

    b.) Suppose you roll two dice. What is the probability of getting a sum of 7, given the first die is a 4?

        Solution:
            Since you know that the first die is a 4, this is your restricted sample space, so
                R = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)}
            Out of this restricted sample space, the way to get a sum of 7 is {(4,3)}. Thus

$$P(\text{sum of } 7|\text{the first die is a } 4) = \frac{1}{6}$$

c.) Suppose you roll two dice. What is the probability of getting the second die a 2, given the sum is a 9?

Solution:
Since you know the sum is a 9, this is your restricted sample space, so
R = {(3,6), (4,5), (5,4), (6,3)}
Out of this restricted sample space there is no way to get the second die a 2. Thus
$$P(\text{second die is a } 2|\text{sum is } 9) = 0$$

d.) Suppose you pick a card from a deck. What is the probability of getting a Spade, given that the card is a Jack?

Solution:
Since you know that the card is a Jack, this is your restricted sample space, so
R = {JS, JC, JD, JH}
Out of this restricted sample space, the way to get a Spade is {JS}. Thus
$$P(\text{Spade}|\text{Jack}) = \frac{1}{4}$$

e.) Suppose you pick a card from a deck. What is the probability of getting an Ace, given the card is a Queen?

Solution:
Since you know that the card is a Queen, then this is your restricted sample space, so
R = {QS, QC, QD, QH}
Out of this restricted sample space, there is no way to get an Ace, thus
$$P(\text{Ace}|\text{Queen}) = 0$$

If you look at the results of example #4.2.7 part d and example #4.3.1 part b, you will notice that you get the same answer. This means that knowing that the first die is a 4 did not change the probability that the sum is a 7. This added knowledge did not help you in any way. It is as if that information was not given at all. However, if you compare example #4.2.7 part b and example #4.3.1 part a, you will notice that they are not the same answer. In this case, knowing that the first die is a 2 did change the probability of getting a sum of 5. In the first case, the events sum of 7 and first die is a 4 are called **independent events.** In the second case, the events sum of 5 and first die is a 2 are called **dependent events.**

Events A and B are considered **independent events** if the fact that one event happens does not change the probability of the other event happening. In other words, events A and B are independent if the fact that B has happened does not affect the probability of event A happening and the fact that A has happened does not affect the probability of event B happening. Otherwise, the two events are dependent.
In symbols, A and B are independent if

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B)$$

Example #4.3.2: Independent Events
    a.) Suppose you roll two dice. Are the events "sum of 7" and "first die is a 3" independent?

        Solution:
            To determine if they are independent, you need to see if $P(A|B) = P(A)$.

            It doesn't matter which event is A or B, so just assign one as A and one as B.
            Let A = sum of 7 = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)} and B = first die is a 3 = {(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)}
            $P(A|B)$ means that you assume that B has happened. The restricted sample space is B,
                R = {(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)}
            In this restricted sample space, the way for A to happen is {(3,4)}, so

$$P(A|B) = \frac{1}{6}$$

            The $P(A) = \frac{6}{36} = \frac{1}{6}$

            $P(A|B) = P(A)$. Thus "sum of 7" and "first die is a 3" are independent events.

    b.) Suppose you roll two dice. Are the events "sum of 6" and "first die is a 4" independent?

        Solution:
            To determine if they are independent, you need to see if $P(A|B) = P(A)$.

            It doesn't matter which event is A or B, so just assign one as A and one as B.
            Let A = sum of 6 = {(1,5), (2,4), (3,3), (4,2), (5,1)} and B = first die is a 4 = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)}, so

$$P(A) = \frac{5}{36}$$

            For $P(A|B)$, the restricted sample space is B,
                R = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)}
            In this restricted sample space, the way for A to happen is {(4,2)}, so

$$P(A|B) = \frac{1}{6}.$$

In this case, "sum of 6" and "first die is a 4" are dependent since $P(A|B) \neq P(A)$.

c.) Suppose you pick a card from a deck. Are the events "Jack" and "Spade" independent?

Solution:

To determine if they are independent, you need to see if $P(A|B) = P(A)$.

It doesn't matter which event is A or B, so just assign one as A and one as B.

Let A = Jack = {JS, JC, JD, JH} and B = Spade {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS}

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

For $P(A|B)$, the restricted sample space is B,

R = {2S, 3S, 4S, 5S, 6S, 7S, 8S, 9S, 10S, JS, QS, KS, AS}

In this restricted sample space, the way A happens is {JS}, so

$$P(A|B) = \frac{1}{13}$$

In this case, "Jack" and "Spade" are independent since $P(A|B) = P(A)$.

d.) Suppose you pick a card from a deck. Are the events "Heart" and "Red" card independent?

Solution:

To determine if they are independent, you need to see if $P(A|B) = P(A)$.

It doesn't matter which event is A or B, so just assign one as A and one as B.

Let A = Heart = {2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AH} and B = Red card = {2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD, AD, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AH}, so

$$P(A) = \frac{13}{52} = \frac{1}{4}$$

For $P(A|B)$, the restricted sample space is B,

R = {2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D, JD, QD, KD, AD, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H, JH, QH, KH, AH}

In this restricted sample space, the way A can happen is 13,

$$P(A|B) = \frac{13}{26} = \frac{1}{2}.$$

In this case, "Heart" and "Red" card are dependent, since $P(A|B) \neq P(A)$.

e.) Suppose you have two children via separate births. Are the events "the first is a boy" and "the second is a girl" independent?

Solution:
In this case, you actually don't need to do any calculations. The gender of one child does not affect the gender of the second child, the events are independent.

f.) Suppose you flip a coin 50 times and get a head every time, what is the probability of getting a head on the next flip?

Solution:
Since one flip of the coin does not affect the next flip (the coin does not remember what it did the time before), the probability of getting a head on the next flip is still one-half.

Multiplication Rule:
Two more useful formulas:
If two events are dependent, then $P(A \text{ and } B) = P(A) * P(B|A)$
If two events are independent, then $P(A \text{ and } B) = P(A) * P(B)$

If you solve the first equation for $P(B|A)$, you obtain $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)}$, which is a

formula to calculate a conditional probability. However, it is easier to find a conditional probability by using the restricted sample space and counting unless the sample space is large.

Example #4.3.3: Multiplication Rule

a.) Suppose you pick three cards from a deck, what is the probability that they are all Queens if the cards are not replaced after they are picked?

Solution:

This sample space is too large to write out, so using the multiplication rule makes sense. Since the cards are not replaced, then the probability will change for the second and third cards. They are dependent events. This means that on the second draw there is one less Queen and one less card, and on the third draw there are two less Queens and 2 less cards.

$P(3 \text{ Queens}) = P(Q \text{ on 1st and Q on 2nd and Q on 3rd})$

$$= P(Q \text{ on 1st}) * P(Q \text{ on 2nd} | Q \text{ on 1st}) * P(Q \text{ on 3rd} | \text{1st and 2nd Q})$$

$$= \frac{4}{52} * \frac{3}{51} * \frac{2}{50}$$

$$= \frac{24}{132600}$$

b.) Suppose you pick three cards from a deck, what is the probability that they are all Queens if the cards are replaced after they are picked and before the next card is picked?

Solution:

Again, the sample space is too large to write out, so using the multiplication rule makes sense. Since the cards are put back, one draw has no affect on the next draw and they are all independent.

$P(3 \text{ Queens}) = P(\text{Queen on 1st and Queen on 2nd and Queen on 3rd})$

$$= P(\text{Queen on 1st}) * P(\text{Queen on 2nd}) * P(\text{Queen on 3rd})$$

$$= \frac{4}{52} * \frac{4}{52} * \frac{4}{52}$$

$$= \left(\frac{4}{52}\right)^3$$

$$= \frac{64}{140608}$$

Example #4.3.4: Application Problem
The World Health Organization (WHO) keeps track of how many incidents of leprosy there are in the world. Using the WHO regions and the World Banks income groups, one can ask if an income level and a WHO region are dependent on each other in terms of predicting where the disease is. Data on leprosy cases in different countries was collected for the year 2011 and a summary is presented in table #4.3.1 ("Leprosy: Number of," 2013).

Table #4.3.1: Number of Leprosy Cases

| | World Bank Income Group | | | | |
WHO Region	High Income	Upper Middle Income	Lower Middle Income	Low Income	Row Total
Americas	174	36028	615	0	36817
Eastern Mediterranean	54	6	1883	604	2547
Europe	10	0	0	0	10
Western Pacific	26	216	3689	1155	5086
Africa	0	39	1986	15928	17953
South-East Asia	0	0	149896	10236	160132
Column Total	264	36289	158069	27923	222545

a.) Find the probability that a person with leprosy is from the Americas.

Solution:
There are 36817 cases of leprosy in the Americas out of 222,545 cases worldwide. So,

$$P(\text{Americas}) = \frac{36817}{222545} \approx 0.165$$

There is about a 16.5% chance that a person with leprosy lives in a country in the Americas.

b.) Find the probability that a person with leprosy is from a high-income country.

Solution:
There are 264 cases of leprosy in high-income countries out of 222,545 cases worldwide. So,

$$P(\text{high-income}) = \frac{264}{222545} \approx 0.0001$$

There is about a 0.1% chance that a person with leprosy lives in a high-income country.

c.) Find the probability that a person with leprosy is from the Americas and a high-income country.

Solution:
There are 174 cases of leprosy in countries in a high-income country in the Americas out the 222,545 cases worldwide. So,

$$P(\text{Americas and high-income}) = \frac{174}{222545} 0.0008$$

There is about a 0.08% chance that a person with leprosy lives in a high-income country in the Americas.

d.) Find the probability that a person with leprosy is from a high-income country, given they are from the Americas.

Solution:
In this case you know that the person is in the Americas. You don't need to consider people from Easter Mediterranean, Europe, Western Pacific, Africa, and South-east Asia. You only need to look at the row with Americas at the start. In that row, look to see how many leprosy cases there are from a high-income country. There are 174 countries out of the 36,817 leprosy cases in the Americas. So,

$$P(\text{high-income}|\text{Americas}) = \frac{174}{36817} \approx 0.0047$$

There is 0.47% chance that a person with leprosy is from a high-income country given that they are from the Americas.

e.) Find the probability that a person with leprosy is from a low-income country.

Solution:
There are 27,923 cases of leprosy in low-income countries out of the 222,545 leprosy cases worldwide. So,

$$P(\text{low-income}) = \frac{27923}{222545} \approx 0.125$$

There is a 12.5% chance that a person with leprosy is from a low-income country.

f.) Find the probability that a person with leprosy is from Africa.

Solution:
There are 17,953 cases of leprosy in Africa out of 222,545 leprosy cases worldwide. So,

$$P(\text{Africa}) = \frac{17953}{222545} \approx 0.081$$

There is an 8.1% chance that a person with leprosy is from Africa.

g.) Find the probability that a person with leprosy is from Africa and a low-income country.

Solution:
There are 15,928 cases of leprosy in low-income countries in Africa out of all the 222,545 leprosy cases worldwide. So,
$$P(\text{Africa and low-income}) = \frac{15928}{222545} \approx 0.072$$
There is a 7.2% chance that a person with leprosy is from a low-income country in Africa.

h.) Find the probability that a person with leprosy is from Africa, given they are from a low-income country.

Solution:
In this case you know that the person with leprosy is from low-income country. You don't need to include the high income, upper-middle income, and lower-middle income country. You only need to consider the column headed by low-income. In that column, there are 15,928 cases of leprosy in Africa out of the 27,923 cases of leprosy in low-income countries. So,
$$P(\text{Africa}|\text{low-income}) = \frac{15928}{27923} \approx 0.570$$
There is a 57.0% chance that a person with leprosy is from Africa, given that they are from a low-income country.

i.) Are the events that a person with leprosy is from "Africa" and "low-income country" independent events? Why or why not?

Solution:
In order for these events to be independent, either
$P(\text{Africa}|\text{low-income}) = P(\text{Africa})$ or $P(\text{low-income}|\text{Africa}) = P(\text{low-income})$
have to be true. Part (h) showed $P(\text{Africa}|\text{low-income}) \approx 0.570$ and part (f) showed $P(\text{Africa}) \approx 0.081$. Since these are not equal, then these two events are dependent.

j.) Are the events that a person with leprosy is from "Americas" and "high-income country" independent events? Why or why not?

Solution:
In order for these events to be independent, either
$P(\text{Americas}|\text{high-income}) = P(\text{Americas})$ or

$P(\text{high-income}|\text{Americas}) = P(\text{high-income})$ have to be true. Part (d) showed

$P(\text{high-income}|\text{Americas}) \approx 0.0047$ and part (b) showed $P(\text{high-income}) \approx 0.001$

Since these are not equal, then these two events are dependent.

A big deal has been made about the difference between dependent and independent events while calculating the probability of *and* compound events. You must multiply the probability of the first event with the conditional probability of the second event.

Why do you care? You need to calculate probabilities when you are performing sampling, as you will learn later. But here is a simplification that can make the calculations a lot easier: when the sample size is very small compared to the population size, you can assume that the conditional probabilities just don't change very much over the sample.

For example, consider acceptance sampling. Suppose there is a big population of parts delivered to you factory, say 12,000 parts. Suppose there are 85 defective parts in the population. You decide to randomly select ten parts, and reject the shipment. What is the probability of rejecting the shipment?

There are many different ways you could reject the shipment. For example, maybe the first three parts are good, one is bad, and the rest are good. Or all ten parts could be bad, or maybe the first five. So many ways to reject! But there is only *one* way that you'd accept the shipment: if *all ten* parts are good. That would happen if the first part is good, **and** the second part is good, **and** the third part is good, and so on. Since the probability of the second part being good is (slightly) dependent on whether the first part was good, technically you should take this into consideration when you calculate the probability that all ten are good.

The probability of getting the first sampled part good is $\dfrac{12000-85}{12000} = \dfrac{11915}{12000}$. So the

probability that all ten being good is $\dfrac{11915}{12000} * \dfrac{11914}{11999} * \dfrac{11913}{11998} * \ldots * \dfrac{11906}{11991} \approx 93.1357\%$.

If instead you assume that the probability doesn't change much, you get

$\left(\dfrac{11915}{12000}\right)^{10} \approx 93.1382\%$. So as you can see, there is not much difference. So here is the

rule: if the sample is very small compared to the size of the population, then you can assume that the probabilities are independent, even though they aren't technically. By the way, the probability of rejecting the shipment is $1 - 0.9314 = 0.0686 = 6.86\%$.

Section 4.3: Homework

1.) Are owning a refrigerator and owning a car independent events? Why or why not?

2.) Are owning a computer or tablet and paying for Internet service independent events? Why or why not?

3.) Are passing your statistics class and passing your biology class independent events? Why or why not?

4.) Are owning a bike and owning a car independent events? Why or why not?

5.) An experiment is picking a card from a fair deck.
   a.) What is the probability of picking a Jack given that the card is a face card?
   b.) What is the probability of picking a heart given that the card is a three?
   c.) What is the probability of picking a red card given that the card is an ace?
   d.) Are the events Jack and face card independent events? Why or why not?
   e.) Are the events red card and ace independent events? Why or why not?

6.) An experiment is rolling two dice.
   a.) What is the probability that the sum is 6 given that the first die is a 5?
   b.) What is the probability that the first die is a 3 given that the sum is 11?
   c.) What is the probability that the sum is 7 given that the fist die is a 2?
   d.) Are the two events sum of 6 and first die is a 5 independent events? Why or why not?
   e.) Are the two events sum of 7 and first die is a 2 independent events? Why or why not?

7.) You flip a coin four times. What is the probability that all four of them are heads?

8.) You flip a coin six times. What is the probability that all six of them are heads?

9.) You pick three cards from a deck with replacing the card each time before picking the next card. What is the probability that all three cards are kings?

10.) You pick three cards from a deck without replacing a card before picking the next card. What is the probability that all three cards are kings?

11.) The number of people who survived the Titanic based on class and sex is in table #4.3.2 ("Encyclopedia Titanica," 2013). Suppose a person is picked at random from the survivors.

Table #4.3.2: Surviving the Titanic

Class	Sex Female	Male	Total
1st	134	59	193
2nd	94	25	119
3rd	80	58	138
Total	308	142	450

a.) What is the probability that a survivor was female?
b.) What is the probability that a survivor was in the 1st class?
c.) What is the probability that a survivor was a female given that the person was in 1st class?
d.) What is the probability that a survivor was a female and in the 1st class?
e.) What is the probability that a survivor was a female or in the 1st class?
f.) Are the events survivor is a female and survivor is in 1st class mutually exclusive? Why or why not?
g.) Are the events survivor is a female and survivor is in 1st class independent? Why or why not?

12.) Researchers watched groups of dolphins off the coast of Ireland in 1998 to determine what activities the dolphins partake in at certain times of the day ("Activities of dolphin," 2013). The numbers in table #4.3.3 represent the number of groups of dolphins that were partaking in an activity at certain times of days.

Table #4.3.3: Dolphin Activity

Activity	Period Morning	Noon	Afternoon	Evening	Total
Travel	6	6	14	13	39
Feed	28	4	0	56	88
Social	38	5	9	10	62
Total	72	15	23	79	189

a.) What is the probability that a dolphin group is partaking in travel?
b.) What is the probability that a dolphin group is around in the morning?
c.) What is the probability that a dolphin group is partaking in travel given that it is morning?
d.) What is the probability that a dolphin group is around in the morning given that it is partaking in socializing?
e.) What is the probability that a dolphin group is around in the afternoon given that it is partaking in feeding?
f.) What is the probability that a dolphin group is around in the afternoon and is partaking in feeding?
g.) What is the probability that a dolphin group is around in the afternoon or is partaking in feeding?
h.) Are the events dolphin group around in the afternoon and dolphin group feeding mutually exclusive events? Why or why not?
i.) Are the events dolphin group around in the morning and dolphin group partaking in travel independent events? Why or why not?

Section 4.4: Counting Techniques

There are times when the sample space or event space are very large, that it isn't feasible to write it out. In that case, it helps to have mathematical tools for counting the size of the sample space and event space. These tools are known as counting techniques.

Multiplication Rule in Counting Techniques

If task 1 can be done m_1 ways, task 2 can be done m_2 ways, and so forth to task n being done m_n ways. Then the number of ways to do task 1, 2,..., n together would be $m_1 * m_2 * \ldots * m_n$.

Example #4.4.1: Multiplication Rule in Counting

A menu offers a choice of 3 salads, 8 main dishes, and 5 desserts. How many different meals consisting of one salad, one main dish, and one dessert are possible?

Solution:

There are three tasks, picking a salad, a main dish, and a dessert. The salad task can be done 3 ways, the main dish task can be done 8 ways, and the dessert task can be done 5 ways. The ways to pick a salad, main dish, and dessert are

$$\underset{\text{salad}}{3} \quad \underset{\text{main}}{8} \quad \underset{\text{dessert}}{5} = 120 \text{ different meals}$$

Example #4.4.2: Multiplication Rule in Counting

How many three letter "words" can be made from the letters a, b, and c with no letters repeating? A "word" is just an ordered group of letters. It doesn't have to be a real word in a dictionary.

Solution:

There are three tasks that must be done in this case. The tasks are to pick the first letter, then the second letter, and then the third letter. The first task can be done 3 ways since there are 3 letters. The second task can be done 2 ways, since the first task took one of the letters. The third task can be done 1 ways, since the first and second task took two of the letters. There are
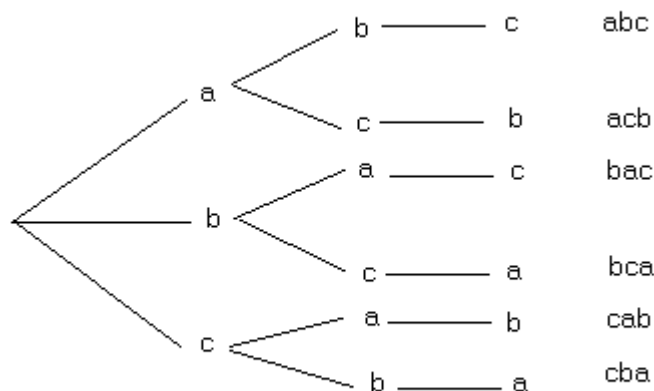
$$\underset{\text{first letter}}{3} * \underset{\text{second letter}}{2} * \underset{\text{third letter}}{1}$$

Which is

$$3 * 2 * 1 = 6$$

You can also look at this in a tree diagram:

So, there are 6 different "words."

In example #4.4.2, the solution was found by find $3*2*1=6$. Many counting problems involve multiplying a list of decreasing numbers. This is called a **factorial**. There is a special symbol for this and a special button on your calculator.

Factorial
$$n!=n(n-1)(n-2)\cdots(3)(2)(1)$$

As an example:
$$5!=5*4*3*2*1=120$$
$$8!=8*7*6*5*4*3*2*1=40320$$

0 factorial is defined to be $0!=1$ and 1 factorial is defined to be $1!=1$.

Sometimes you are trying to select *r* objects from *n* total objects. The number of ways to do this depends on if the order you choose the *r* objects matters or if it doesn't. As an example if you are trying to call a person on the phone, you have to have their number in the right order. Otherwise, you call someone you didn't mean to. In this case, the order of the numbers matters. If however you were picking random numbers for the lottery, it doesn't matter which number you pick first. As long as you have the same numbers that the lottery people pick, you win. In this case the order doesn't matter. A **permutation** is an arrangement of items with a specific order. You use permutations to count items when the order matters. When the order doesn't matter you use combinations. A **combination** is an arrangement of items when order is not important. When you do a counting problem, the first thing you should ask yourself is "does order matter?"

Permutation Formula
Picking *r* objects from *n* total objects when order matters
$$_nP_r = \frac{n!}{(n-r)!}$$

Combination Formula
Picking *r* objects from *n* total objects when order doesn't matter
$$_nC_r = \frac{n!}{r!(n-r)!}$$

Example #4.4.3: Calculating the Number of Ways
In a club with 15 members, how many ways can a slate of 3 officers consisting of a president, vice-president, and secretary/treasurer be chosen?

Solution:
In this case the order matters. If you pick person 1 for president, person 2 for vice-president, and person 3 for secretary/treasurer you would have different officers than if you picked person 2 for president, person 1 for vice-president, and person 3 for secretary/treasurer. This is a permutation problem with $n = 15$ and $r = 3$.
$$_{15}P_3 = \frac{15!}{(15-3)!} = \frac{15!}{12!} = 2730$$

Example #4.4.4: Calculating the Number of Ways
Suppose you want to pick 7 people out of 20 people to take part in a survey. How many ways can you do this?

Solution:
In this case the order doesn't matter, since you just want 7 people. This is a combination with $n = 20$ and $r = 7$.
$$_{20}C_7 = \frac{20!}{7!(20-7)!} = \frac{20!}{7!13!} = 77520$$

Most calculators have a factorial button on them, and many have the combination and permutation functions also. R has a combination command.

Section 4.4: Homework

1.) You are going to a benefit dinner, and need to decide before the dinner what you want for salad, main dish, and dessert. You have 2 different salads to choose from, 3 main dishes, and 5 desserts. How many different meals are available?

2.) How many different phone numbers are possible in the area code 928?

3.) You are opening a T-shirt store. You can have long sleeves or short sleeves, three different colors, five different designs, and four different sizes. How many different shirts can you make?

4.) The California license plate has one number followed by three letters followed by three numbers. How many different license plates are there?

5.) Find $_9P_4$

6.) Find $_{10}P_6$

7.) Find $_{10}C_5$

8.) Find $_{20}C_4$

9.) You have a group of twelve people. You need to pick a president, treasurer, and secretary from the twelve. How many different ways can you do this?

10.) A baseball team has a 25-man roster. A batting order has nine people. How many different batting orders are there?

11.) An urn contains five red balls, seven yellow balls, and eight white balls. How many different ways can you pick two red balls?

12.) How many ways can you choose seven people from a group of twenty?

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