Chapter 4: Growth

Population growth is a current topic in the media today. The world population is growing by over 70 million people every year. Predicting populations in the future can have an impact on how countries plan to manage resources for more people. The tools needed to help make predictions about future populations are growth models like the exponential function. This chapter will discuss real world phenomena, like population growth and radioactive decay, using three different growth models.

The growth functions to be examined are linear, exponential, and logistic growth models. Each type of model will be used when data behaves in a specific way and for different types of scenarios. Data that grows by the same amount in each iteration uses a different model than data that increases by a percentage.

Section 4.1: Linear Growth

Starting at the age of 25, imagine if you could save $20 per week, every week, until you retire, how much money would you have stuffed under your mattress at age 65? To solve this problem, we could use a linear growth model. Linear growth has the characteristic of growing by the same amount in each unit of time. In this example, there is an increase of $20 per week; a constant amount is placed under the mattress in the same unit of time.

If we start with $0 under the mattress, then at the end of the first year we would have $20 \times 52 = \$1040$. So, this means you could add $1040 under your mattress every year. At the end of 40 years, you would have $\$1040 \times 40 = \$41,600$ for retirement. This is not the best way to save money, but we can see that it is calculated in a systematic way.

> **Linear Growth:** A quantity grows linearly if it grows by a constant amount for each unit of time.

**Example 4.1.1: City Growth**

> Suppose in Flagstaff Arizona, the number of residents increased by 1000 people per year. If the initial population was 46,080 in 1990, can you predict the population in 2013? This is an example of linear growth because the population grows by a constant amount. We list the population in future years below by adding 1000 people for each passing year.

Table 4.1.1: Table of City Population Growth

	1990	1991	1992	1993	1994	1995	1996
Year	0	1	2	3	4	5	6
Population	46,080	47,080	48,080	49,080	50,080	51,080	52,080

Graph 4.1.2: Graph of Linear Population Growth
This is the graph of the population growth over a six year period in Flagstaff, Arizona. It is a straight line and can be modeled with a linear growth model.
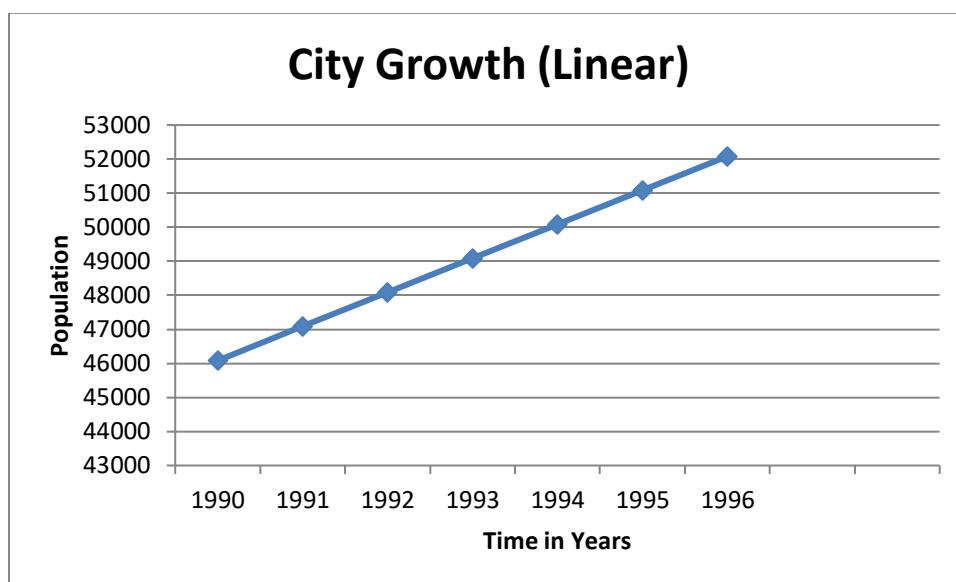
The population growth can be modeled with a linear equation. The initial population P_0 is 48,080. The future population depends on the number of years, t, after the initial year. The model is $P(t) = 46,080 + 1000t$

To predict the population in 2013, we identify how many years it has been from 1990, which is year zero. So $n = 23$ for the year 2013.

$$P(23) = 46,080 + 1000(23) = 69,080$$

The population of Flagstaff in 2013 would be 69,080 people.

Linear Growth Model: Linear growth begins with an initial population called P_0. In each time period or generation t, the population changes by a constant amount called the common difference d. The basic model is:
$P(t) = P_0 + td$

Example 4.1.2: Antique Frog Collection

Dora has inherited a collection of 30 antique frogs. Each year she vows to buy two frogs a month to grow the collection. This is an additional 24 frogs per year. How many frogs will she have is six years? How long will it take her to reach 510 frogs?

The initial population is $P_0 = 30$ and the common difference is $d = 24$. The linear growth model for this problem is:

$P(t) = 30 + 24t$

The first question asks how many frogs Dora will have in six years so, $t = 6$.

$P(6) = 30 + 24(6) = 30 + 144 = 174$ frogs.

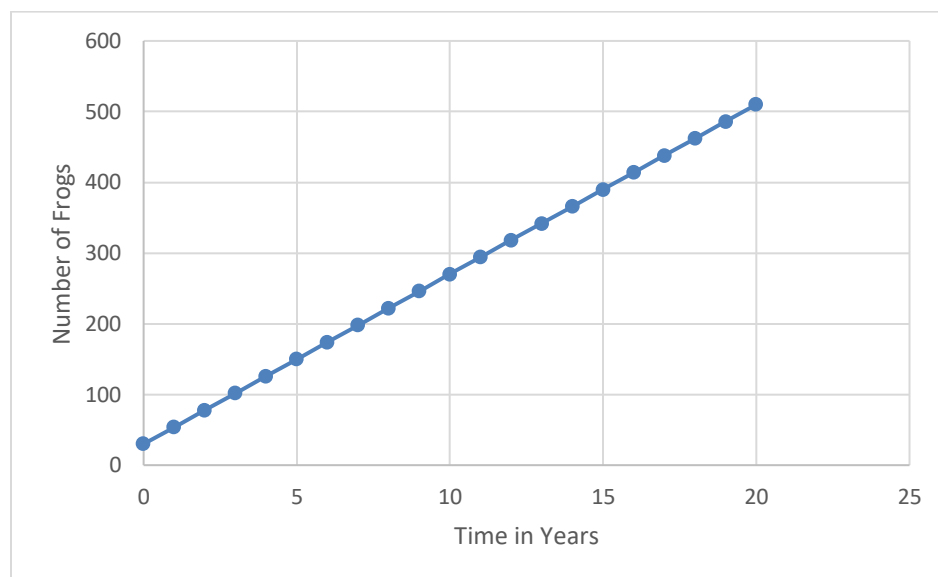The second question asks for the time it will take for Dora to collect 510 frogs. So, $P(t) = 510$ and we will solve for t.

$510 = 30 + 24t$

$480 = 24t$

$20 = t$

It will take 20 years to collect 510 antique frogs.

Graph 4.1.3: Graph of Antique Frog Collection

Note: The graph of the number of antique frogs Dora accumulates over time follows a straight line.

Example 4.1.3: Car Depreciation

Assume a car depreciates by the same amount each year. Joe purchased a car in 2010 for $16,800. In 2014 it is worth $12,000. Find the linear growth model. Predict how much the car will be worth in 2020.

$P_0 = 16,800$ and $P(4) = 12,000$

To find the linear growth model for this problem, we need to find the common difference d.

$$P(t) = P_0 + td$$

$$12,000 = 16,800 + 4d$$

$$-4800 = 4d$$

$$-1200 = d$$

The common difference of depreciation each year is $d = \$-1200$. Thus, the linear growth model for this problem is: $P(t) = 16,800 - 1200t$

Now, to find out how much the car will be worth in 2020, we need to know how many years that is from the purchase year. Since it is ten years later, $t = 10$.

$$P(10) = 16,800 - 1200(10) = 16,800 - 12,000 = 4,800$$

The car is worth $4800 in 2020.

Graph 4.1.4: Graph of Car Value Depreciation
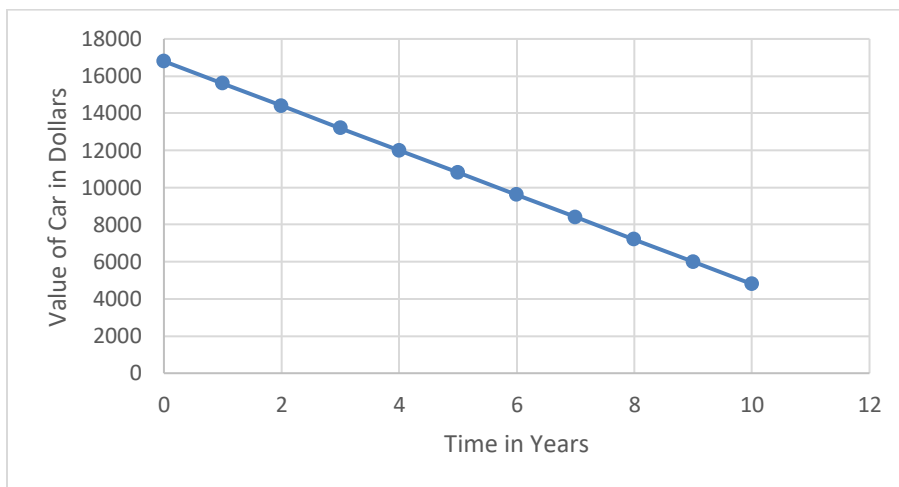
*Note: The value of the car over time follows a decreasing straight line.*

Section 4.2: Exponential Growth

The next growth we will examine is exponential growth. Linear growth occur by adding the same amount in each unit of time. Exponential growth happens when an initial population increases by the same percentage or factor over equal time increments or generations. This is known as relative growth and is usually expressed as percentage. For example, let's say a population is growing by 1.6% each year. For every 1000 people in the population, there will be $1000 \times 0.016 = 16$ more people added per year.

> **Exponential Growth:** A quantity grows exponentially if it grows by a constant factor or rate for each unit of time.

Graph 4.2.1: Graphical Comparison of Linear and Exponential Growth
In this graph, the blue straight line represents linear growth, and the red curved line represents exponential growth.
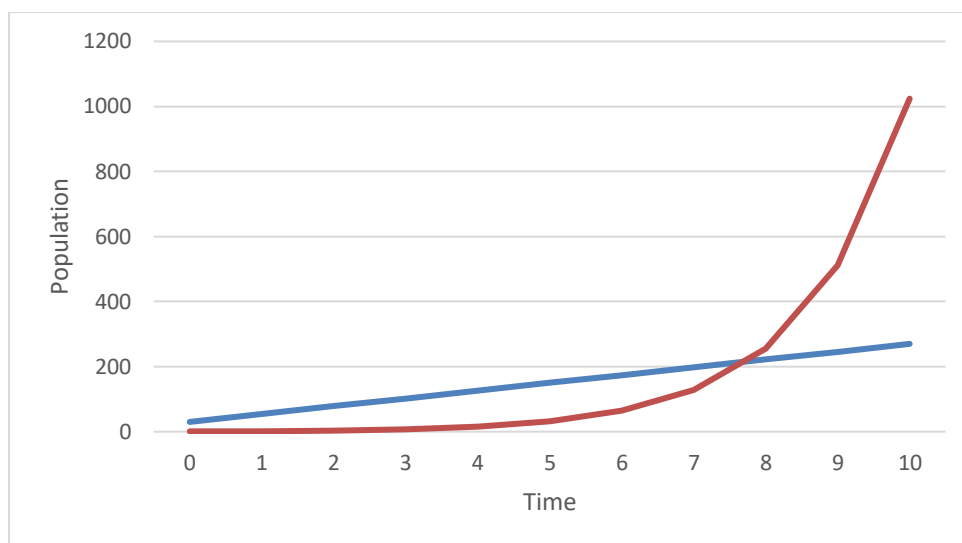
Example 4.2.1: City Growth

A city is growing at a rate of 1.6% per year. The initial population in 2010 is $P_0 = 125,000$. Calculate the city's population over the next few years.

The relative growth rate is 1.6%. This means an additional 1.6% is added on to 100% of the population that already exists each year. This is a factor of 101.6%.

Population in 2011 $= 125,000(1.016)^1 = 127,000$

Population in 2012 $= 127,000(1.016) = 125,000(1.016)^2 = 129,032$

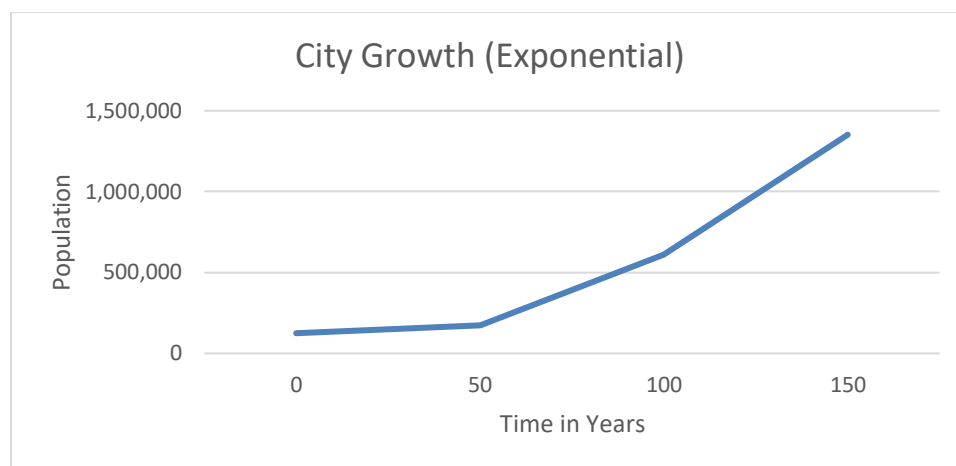Population in 2013 = 129,032(1.016) = 125,000(1.016)³ = 131,097

We can create an equation for the city's growth. Each year the population is 101.6% more than the previous year.

$$P(t) = 125,000(1+0.016)^t$$

Graph 4.2.2: Graph of City Growth

Note: The graph of the city growth follows an exponential growth model.

Example 4.2.2: A Shrinking Population

St. Louis, Missouri has declined in population at a rate of 1.6 % per year over the last 60 years. The population in 1950 was 857,000. Find the population in 2014. (Wikipedia, n.d.)

$$P_0 = 857,000$$

The relative growth rate is 1.6%. This means 1.6% of the population is subtracted from 100% of the population that already exists each year. This is a factor of 98.4%.

Population in 1951 = 857,000(0.984)¹ = 843,288

Population in 1952 = 843,228(0.984) = 857,000(0.984)² = 829,795

Population in 1953 = 828,795(0.984) = 857,000(0.984)³ = 816,519

We can create an equation for the city's growth. Each year the population is 1.6% less than the previous year. $P(t) = 857,000(1-0.016)^t$

So the population of St. Louis Missouri in 2014, when $t = 64$, is:

$$P(64) = 857,000(1 - 0.016)^{64}$$
$$= 857,000(0.984)^{64}$$
$$= 305,258$$

Graph 4.2.3: Graph of St. Louis, Missouri Population Decline
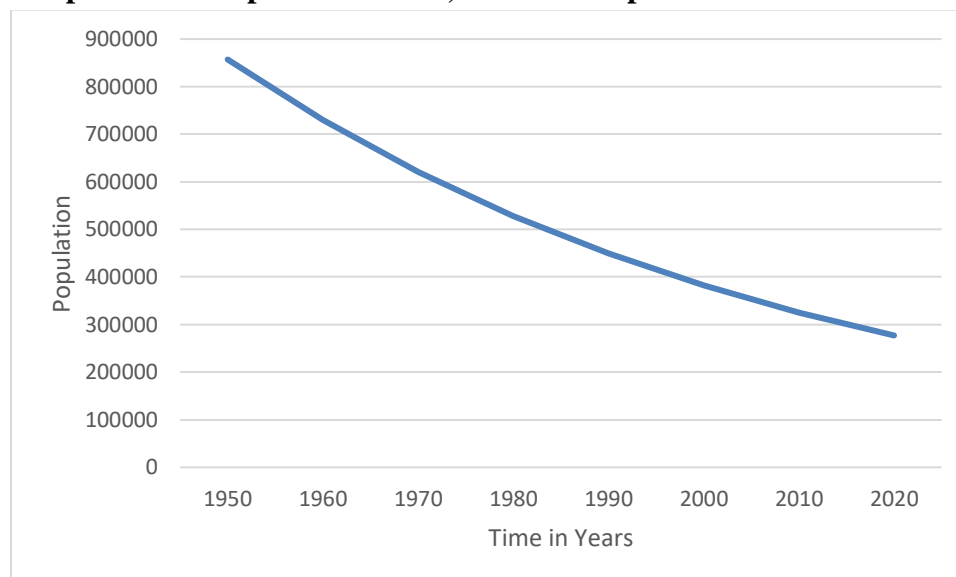
Note: The graph of the population of St. Louis, Missouri over time follows a declining exponential growth model.

Exponential Growth Model $P(t) = P_0(1+r)^t$

P_0 is the initial population,

r is the relative growth rate.

t is the time unit.

r is positive if the population is increasing and negative if the population is decreasing

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Example 4.2.3: Inflation

The average inflation rate of the U.S. dollar over the last five years is 1.7% per year. If a new car cost $18,000 five years ago, how much would it cost today? (U.S. Inflation Calculator, n.d.)

To solve this problem, we use the exponential growth model with $r = 1.7\%$.

$P_0 = 18,000$ and $t = 5$

$$P(t) = 18,000(1+0.017)^t$$

$$P(5) = 18,000(1+0.017)^5 = 19,582.91$$

This car would cost $19,582.91 today.

Example 4.2.4: Ebola Epidemic in Sierra Leone

In May of 2014 there were 15 cases of Ebola in Sierra Leone. By August, there were 850 cases. If the virus is spreading at the same rate (exponential growth), how many cases will there be in February of 2015? (McKenna, 2014)

To solve this problem, we have to find three things; the growth rate per month, the exponential growth model, and the number of cases of Ebola in February 2015. First calculate the growth rate per month. To do this, use the initial population $P_0 = 15$, in May 2014. Also, in August, three months later, the number of cases was 850 so, $P(3) = 850$.

Use these values and the exponential growth model to solve for r.

$$P(t) = P_0(1+r)^t$$

$$850 = 15(1+r)^3$$

$$56.67 = (1+r)^3$$

$$\sqrt[3]{56.67} = \sqrt[3]{(1+r)^3}$$

$$3.84 = 1+r$$

$$2.84 = r$$

The growth rate is 284% per month. Thus, the exponential growth model is:

$$P(t) = 15(1+2.84)^t = 15(3.84)^t$$

Now, we use this to calculate the number of cases of Ebola in Sierra Leone in February 2015, which is nine months after the initial outbreak so, $t = 9$.

$$P(9) = 15(3.84)^9 = 2,725,250$$

If this same exponential growth rate continues, the number of Ebola cases in Sierra Leone in February 2015 would be 2,725,250.
This is a bleak prediction for the community of Sierra Leone. Fortunately, the growth rate of this deadly virus should be reduced by the world community and World Health Organization by providing the needed means to fight the initial spread.

Graph 4.2.4: Graph of Ebola Virus is Sierra Leone
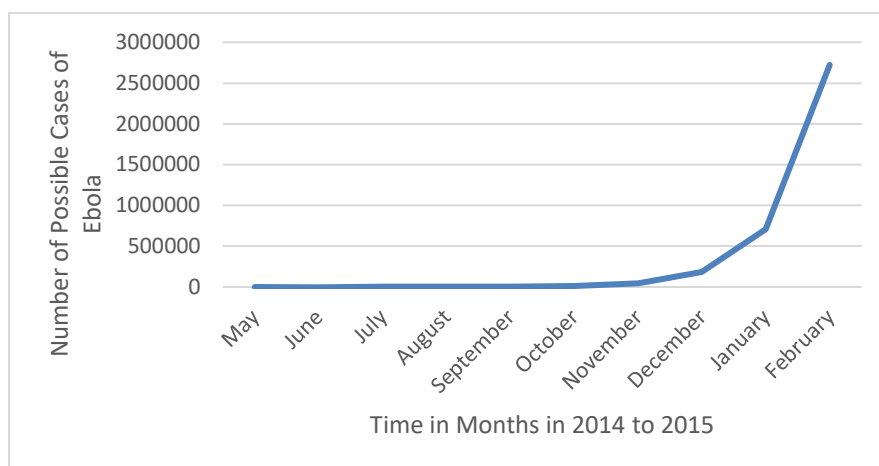
Note: The graph of the number of possible Ebola cases in Sierra Leone over time follows an increasing exponential growth model.

Example 4.2.5: Population Decline in Puerto Rico

According to a new forecast, the population of Puerto Rico is in decline. If the population in 2010 is 3,978,000 and the prediction for the population in 2050 is 3,697,000, find the annual percent decrease rate. (Bloomberg Businessweek, n.d.) To solve this problem, we use the exponential growth model. We need to solve for *r*.

$P(t) = P_0 (1+r)^t$ where t $= 40$ years

$P(40) = 3,697,000$ and $P_0 = 3,978,000$

$3,697,000 = 3,978,000(1+r)^{40}$

$0.92936 = (1+r)^{40}$

$\sqrt[40]{0.92936} = \sqrt[40]{(1+r)^{40}}$

$0.99817 = 1+r$

$-0.0018 = r$

The annual percent decrease is 0.18%.

Section 4.3: Special Cases: Doubling Time and Half-Life

Example 4.3.1: April Fool's Joke

Let's say that on April 1st I say I will give you a penny, on April 2nd two pennies, four pennies on April 3rd, and that I will double the amount each day until the end of the month. How much money would I have agreed to give you on April 30?

With $P_0 = \$0.01$, we get the following table:

Table 4.3.1: April Fool's Joke

Day	Dollar Amount
April 1 = P_0	0.01
April 2 = P_1	0.02
April 3 = P_2	0.04
April 4 = P_3	0.08
April 5 = P_4	0.16
April 6 = P_5	0.32
….	….
April 30 = P_{29}	?

In this example, the money received each day is 100% more than the previous day. If we use the exponential growth model $P(t) = P_0(1+r)^t$ with $r = 1$, we get the doubling time model.

$P(t) = P_0(1+1)^t = P_0(2)^t$

We use it to find the dollar amount when $t = 29$ which represents April 30

$P(29) = 0.01(2)^{29} = \$5,368,709.12$

Surprised?

That is a lot of pennies.

Doubling Time Model:

Example 4.3.2: E. coli Bacteria

A water tank up on the San Francisco Peaks is contaminated with a colony of 80,000 E. coli bacteria. The population doubles every five days. We want to find a model for the population of bacteria present after t days. The amount of time it takes the population to double is five days, so this is our time unit. After t days have passed, then $\frac{t}{5}$ is the number of time units that have passed. Starting with the initial amount of 80,000 bacteria, our doubling model becomes:

$$P(t) = 80,000(2)^{\frac{t}{5}}$$

Using this model, how large is the colony in two weeks' time? We have to be careful that the units on the times are the same; 2 weeks = 14 days.

$$P(14) = 80,000(2)^{\frac{14}{5}} = 557,152$$

The colony is now 557,152 bacteria.

Doubling Time Model: If D is the doubling time of a quantity (the amount of time it takes the quantity to double) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0(2)^{\frac{t}{D}}$

Example 4.3.3: Flies

The doubling time of a population of flies is eight days. If there are initially 100 flies, how many flies will there be in 17 days? To solve this problem, use the doubling time model with $D = 8$ and $P_0 = 100$ so the doubling time model for this problem is:
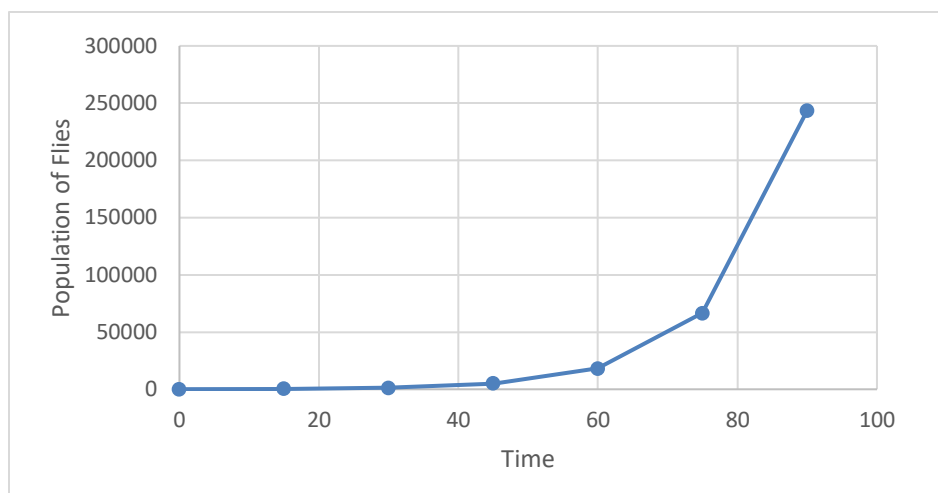
$$P(t) = 100(2)^{\frac{t}{8}}$$

When t = 17 days,

$$P(17) = 100(2)^{\frac{17}{8}} = 436$$

There are 436 flies after 17 days.

Graph 4.3.2: Graph of Fly Population in Three Months

Note: The population of flies follows an exponential growth model.

Sometimes we want to solve for the length of time it takes for a certain population to grow given their doubling time. To solve for the exponent, we use the *log* button on the calculator.

Example 4.3.4: Bacteria Growth

Suppose that a bacteria population doubles every six hours. If the initial population is 4000 individuals, how many hours would it take the population to increase to 25,000?

$P_0 = 4000$ and $D=6$, so the doubling time model for this problem is:

$P(t) = 4000(2)^{\frac{t}{6}}$ Now, find t when $P(t) = 25,000$

$25,000 = 4000(2)^{\frac{t}{6}}$

$\dfrac{25,000}{4000} = \dfrac{\cancel{4000}(2)^{\frac{t}{6}}}{\cancel{4000}}$

$6.25 = (2)^{\frac{t}{6}}$ Now, take the log of both sides of the equation.

$\log 6.25 = \log(2)^{\frac{t}{6}}$ The exponent comes down using rules of logarithms.

$\log 6.25 = \left(\dfrac{t}{6}\right)\log 2$ Now, calculate $\log 6.25$ and $\log 2$ with your calculator.

$0.7959 = \left(\dfrac{t}{6}\right) \times 0.3010$

$\dfrac{0.7959}{0.3010} = \dfrac{t}{6}$

$2.644 = \dfrac{t}{6}$

$t = 15.9$

The population would increase to 25,000 bacteria in approximately 15.9 hours.

Rule of 70:

There is a simple formula for approximating the doubling time of a population. It is called the rule of 70 and it is an approximation for growth rates less than 15%. Do not use this formula if the growth rate is 15% or greater.

Rule of 70: For a quantity growing at a constant percentage rate (not written as a decimal), *R*, per time period, the doubling time is approximately given by

Doubling time $D \approx \dfrac{70}{R}$

Example 4.3.5: Bird Population

A bird population on a certain island has an annual growth rate of 2.5% per year. Approximate the number of years it will take the population to double. If the initial population is 20 birds, use it to find the bird population of the island in 17 years.

To solve this problem, first approximate the population doubling time.

Doubling time $D \approx \dfrac{70}{2.5} = 28$ years.

With the bird population doubling in 28 years, we use the doubling time model to find the population is 17 years.

$P(t) = 20(2)^{\frac{t}{28}}$

When $t = 17$ years

$P(16) = 20(2)^{\frac{17}{28}} = 30.46$

There will be 30 birds on the island in 17 years.

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Example 4.3.6: Cancer Growth Rate

A certain cancerous tumor doubles in size every six months. If the initial size of the tumor is four cells, how many cells will there in three years? In seven years?

To calculate the number of cells in the tumor, we use the doubling time model. Change the time units to be the same. The doubling time is six months = 0.5 years.

$$P(t) = 4(2)^{\frac{t}{0.5}}$$

When $t = 3$ years

$$P(3) = 4(2)^{\frac{3}{0.5}} = 256 \text{ cells}$$

When $t = 7$ years

$$P(7) = 4(2)^{\frac{7}{0.5}} = 65,536 \text{ cells}$$

Example 4.3.7: Approximating Annual Growth Rate

Suppose that a certain city's population doubles every 12 years. What is the approximate annual growth rate of the city?

By solving the doubling time model for the growth rate, we can solve this problem.

$$D \approx \frac{70}{R}$$

$$R \cdot D \approx \frac{70}{\cancel{R}} \cdot \cancel{R}$$

$$RD \approx 70$$

$$\frac{R\cancel{D}}{\cancel{D}} \approx \frac{70}{D}$$

Annual growth rate $R \approx \dfrac{70}{D}$

$$R = \frac{70}{12} = 5.83\%$$

The annual growth rate of the city is approximately 5.83%

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Exponential Decay and Half-Life Model:

The half-life of a material is the time it takes for a quantity of material to be cut in half. This term is commonly used when describing radioactive metals like uranium or plutonium. For example, the half-life of carbon-14 is 5730 years.

If a substance has a half-life, this means that half of the substance will be gone in a unit of time. In other words, the amount decreases by 50% per unit of time. Using the exponential growth model with a decrease of 50%, we have

$$P(t) = P_0(1-0.5)^t = P_0(0.5)^t = P_0\left(\frac{1}{2}\right)^t$$

Example 4.3.8: Half-Life

Let's say a substance has a half-life of eight days. If there are 40 grams present now, how much is left after three days? We want to find a model for the quantity of the substance that remains after t days. The amount of time it takes the quantity to be reduced by half is eight days, so this is our time unit. After t days have passed, then $\frac{t}{8}$ is the number of time units that have passed. Starting with the initial amount of 40, our half-life model becomes:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{8}}$$

With $t = 3$

$$P(3) = 40\left(\frac{1}{2}\right)^{\frac{3}{8}} = 30.8$$

There are 30.8 grams of the substance remaining after three days.

Half-Life Model: If H is the half-life of a quantity (the amount of time it takes the quantity be cut in half) and P_0 is the initial amount of the quantity then the amount of the quantity present after t units of time is $P(t) = P_0\left(\frac{1}{2}\right)^{\frac{t}{H}}$

Example 4.3.9: Lead-209

Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 40 milligrams of this isotope is created in an experiment, how much is left after 14 hours? Use the half-life model to solve this problem.
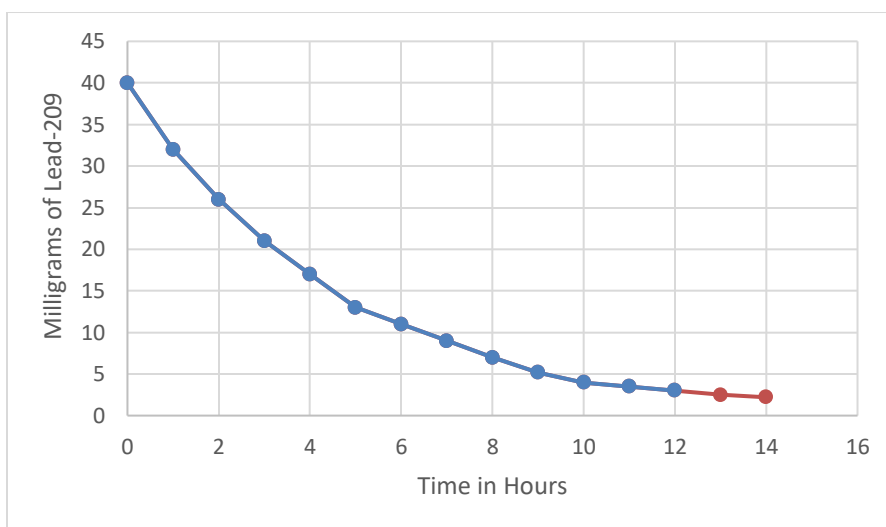
$P_0 = 40$ and $H = 3.3$, so the half-life model for this problem is:

$$P(t) = 40\left(\frac{1}{2}\right)^{\frac{t}{3.3}} \text{ when } t = 14 \text{ hours,}$$

$$P(14) = 40\left(\frac{1}{2}\right)^{\frac{14}{3.3}} = 2.1$$

There are 2.1 milligrams of Lead-209 remaining after 14 hours.

Graph 4.3.3: Lead-209 Decay Graph

Note: The milligrams of Lead-209 remaining follows a decreasing exponential growth model.

Example 4.3.10: Nobelium-259

Nobelium-259 has a half-life of 58 minutes. If you have 1000 grams, how much will be left in two hours? We solve this problem using the half-life model. Before we begin, it is important to note the time units. The half-life is given in minutes and we want to know how much is left in two hours. Convert hours to minutes when using the model: two hours = 120 minutes.

$P_0 = 1000$ and $H = 58$ minutes, so the half-life model for this problem is:

$$P(t) = 1000 \left(\frac{1}{2} \right)^{\frac{t}{58}}$$

When $t = 120$ minutes,

$$P(120) = 1000 \left(\frac{1}{2} \right)^{\frac{120}{58}} = 238.33$$

There are 238 grams of Nobelium-259 is remaining after two hours.

Example 4.3.11: Carbon-14

Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 1947, earthenware jars containing what are known as the Dead Sea Scrolls were found. Analysis indicated that the scroll wrappings contained 76% of their original carbon-14. Estimate the age of the Dead Sea Scrolls. In this problem, we want to estimate the age of the scrolls. In 1947, 76% of the carbon-14 remained. This means that the amount remaining at time t divided by the original amount of carbon-14, P_0, is equal to 76%. So, $\dfrac{P(t)}{P_0} = 0.76$ we use this fact to solve for t.

$$P(t) = P_0 \left(\frac{1}{2} \right)^{\frac{t}{5730}}$$

$$\frac{P(t)}{P_0} = \left(\frac{1}{2} \right)^{\frac{t}{5730}}$$

$0.76 = \left(\dfrac{1}{2} \right)^{\frac{t}{5730}}$ Now, take the log of both sides of the equation.

$\log 0.76 = \log \left(\dfrac{1}{2} \right)^{\frac{t}{5730}}$ The exponent comes down using rules of logarithms.

$\log 0.76 = \left(\dfrac{t}{5730} \right) \log \dfrac{1}{2}$ Now, calculate $\log 0.76$ and $\log \dfrac{1}{2}$ with your calculator.

$$-0.1192 = \left(\frac{t}{5730} \right) \times (-0.3010)$$

$$\frac{-0.1192}{-0.3010} = \frac{t}{5730}$$

$0.3960 = \dfrac{t}{5730}$ $t = 2269.08$. The Dead Sea Scrolls are well over 2000 years old.

Example 4.3.12: Plutonium

Plutonium has a half-life of 24,000 years. Suppose that 50 pounds of it was dumped at a nuclear waste site. How long would it take for it to decay into 10 lbs?

$P_0 = 50$ and $H = 24,000$, so the half-life model for this problem is:

$$P(t) = 50\left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

Now, find t when $P(t) = 10$.

$$10 = 50\left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

$$\frac{10}{50} = \left(\frac{1}{2}\right)^{\frac{t}{24,000}}$$

$$0.2 = \left(\frac{1}{2}\right)^{\frac{t}{24,000}} \quad \text{Now, take the log of both sides of the equation.}$$

$$\log 0.2 = \log\left(\frac{1}{2}\right)^{\frac{t}{24,000}} \quad \text{The exponent comes down using rules of logarithms.}$$

$$\log 0.2 = \left(\frac{t}{24,000}\right)\log\frac{1}{2} \quad \text{Now, calculate } \log 0.2 \text{ and } \log\frac{1}{2} \text{ with your calculator.}$$

$$-0.6990 = \left(\frac{t}{24,000}\right)\times(-0.3010)$$

$$\frac{-0.6990}{-0.3010} = \frac{t}{24,000}$$

$$2.322 = \frac{t}{24,000}$$

$$t = 55,728$$

The quantity of plutonium would decrease to 10 pounds in approximately 55,728 years.

Rule of 70 for Half-Life:

There is simple formula for approximating the half-life of a population. It is called the rule of 70 and is an approximation for decay rates less than 15%. Do not use this formula if the decay rate is 15% or greater.

Rule of 70: For a quantity decreasing at a constant percentage (not written as a decimal), R, per time period, the half-life is approximately given by:

Half-life $H \approx \dfrac{70}{R}$

Example 4.3.13: Elephant Population

The population of wild elephants is decreasing by 7% per year. Approximate the half -life for this population. If there are currently 8000 elephants left in the wild, how many will remain in 25 years?

To solve this problem, use the half-life approximation formula.

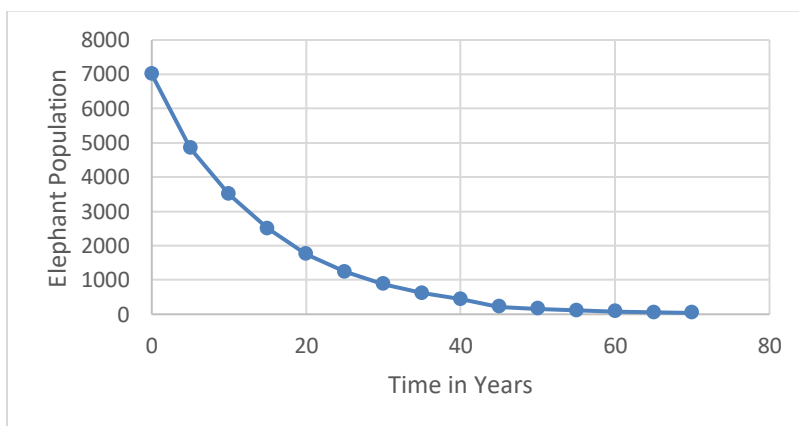Half-life $H \approx \dfrac{70}{7} = 10$ years

$P_0 = 7000$, $H = 10$ years, so the half-life model for this problem is:

$$P(t) = 7000 \left(\frac{1}{2} \right)^{\frac{t}{10}}$$

When $t = 25$,

$$P(25) = 7000 \left(\frac{1}{2} \right)^{\frac{25}{10}} = 1237.4$$

There will be approximately 1237 wild elephants left in 25 years.

Graph 4.3.4: Elephant Population over a 70 Year Span.

Note: The population of elephants follows a decreasing exponential growth model.

Table 4.3.5: Review of Exponents and Log Rules

<table>
<tr><th colspan="2">Review of Exponent Rules and Logarithm Rules</th></tr>
<tr><th>Rules of Exponents</th><th>Rules of Logarithm for the Common Logarithm (Base 10)</th></tr>
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Definition of an Exponent

$a^n = a \cdot a \cdot a \cdot a \cdot ..... \cdot a$

(*n* *a*'s multiplied together)

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Definition of a Logarithm

$10^y = x$ if and only if $\log x = y$

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Zero Rule $\qquad\qquad a^0 = 1$

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Product Rule $\qquad a^m \cdot a^n = a^{m+n}$

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Product Rule $\qquad \log(xy) = \log x + \log y$

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Quotient Rule $\qquad \dfrac{a^m}{a^n} = a^{m-n}$

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Quotient Rule $\qquad \log\left(\dfrac{x}{y}\right) = \log x - \log y$

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Power Rule $\qquad \left(a^n\right)^m = a^{n \cdot m}$

</td>
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Power Rule $\qquad \log x^r = r \log x \quad (x > 0)$

</td>
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Distributive Rules

$(ab)^n = a^n b^n, \ \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

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$\log 10^x = x \log 10 = x$

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Negative Exponent Rules

$a^{-n} = \dfrac{1}{a^n}, \ \left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

</td>
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$10^{\log x} = x \quad (x > 0)$

</td>
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</table>

Section 4.4: Natural Growth and Logistic Growth

In this chapter, we have been looking at linear and exponential growth. Another very useful tool for modeling population growth is the natural growth model. This model uses base *e*, an irrational number, as the base of the exponent instead of $(1 + r)$. You may remember learning about *e* in a previous class, as an exponential function and the base of the natural logarithm.

The Natural Growth Model: $P(t) = P_0 e^{kt}$ where P_0 is the initial population, *k* is the growth rate per unit of time, and *t* is the number of time periods.
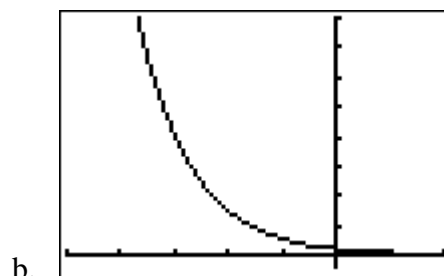

Given $P_0 > 0$, if $k > 0$, this is an exponential growth model, if $k < 0$, this is an exponential decay model.

Figure 4.4.1: Natural Growth and Decay Graphs

a. Natural growth function $P(t) = e^t$
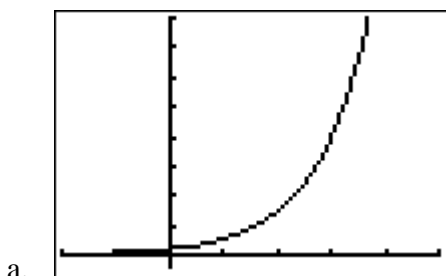
b. Natural decay function $P(t) = e^{-t}$

a.

b.

Example 4.4.1: Drugs in the Bloodstream

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after t hours is given by the model $P(t) = 40e^{-.25t}$

How many milligrams are in the blood after two hours?

To solve this problem, we use the given equation with $t = 2$

$P(2) = 40e^{-0.25(2)}$
$P(2) = 24.26$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

In the next example, we can see that the exponential growth model does not reflect an accurate picture of population growth for natural populations.

Example 4.4.2: Ants in the Yard

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his back yard, with a population of about 100 ants. If conditions are just right red ant colonies have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

We solve this problem using the natural growth model.

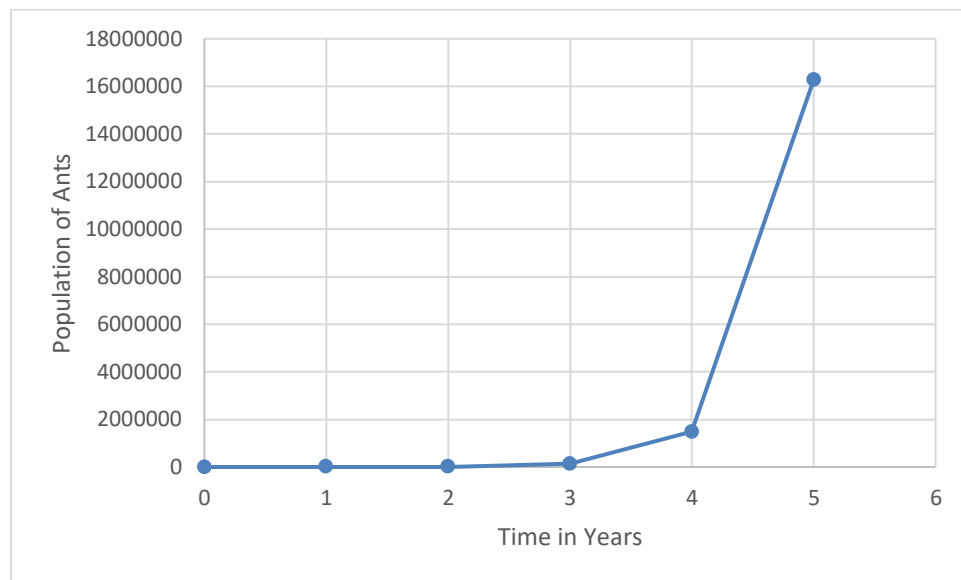$$P(t) = 100e^{2.4t}$$

In one year, $t = 1$, we have

$$P(1) = 100e^{2.4(1)} = 1102 \text{ ants}$$

In five years, $t = 5$, we have

$$P(5) = 100e^{2.4(5)} = 16,275,479 \text{ ants}$$

That is a lot of ants! Bob will not let this happen in his back yard!

Graph 4.4.2: Graph of Ant Population Growth in Bob's Yard.

Note: The population of ants in Bob's back yard follows an exponential (or natural) growth model.

The problem with exponential growth is that the population grows without bound and, at some point, the model will no longer predict what is happening since the number of resources available is limited. Populations cannot continue to grow on a purely physical level, eventually death occurs, and a limiting population is reached.

Another growth model for living organisms in the logistic growth model. The logistic growth model has a maximum population called the carrying capacity. As the population grows, the number of individuals in the population grows to the carrying capacity and stays there. This is the maximum population the environment can sustain.


Logistic Growth Model: $P(t) = \dfrac{M}{1 + ke^{-ct}}$ where M, c, and k are positive constants and t is the number of time periods.

Figure 4.4.3: Comparison of Exponential Growth and Logistic Growth

The horizontal line K on this graph illustrates the carrying capacity. *However, this book uses M to represent the carrying capacity rather than K.*
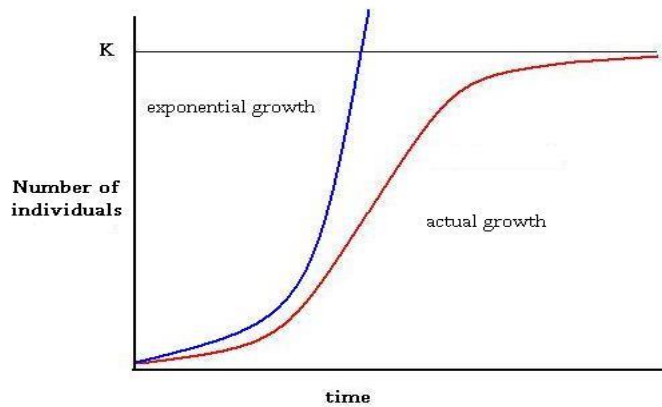
(Logistic Growth Image 1, n.d.)

Figure 4.4.4: Logistic Growth Model
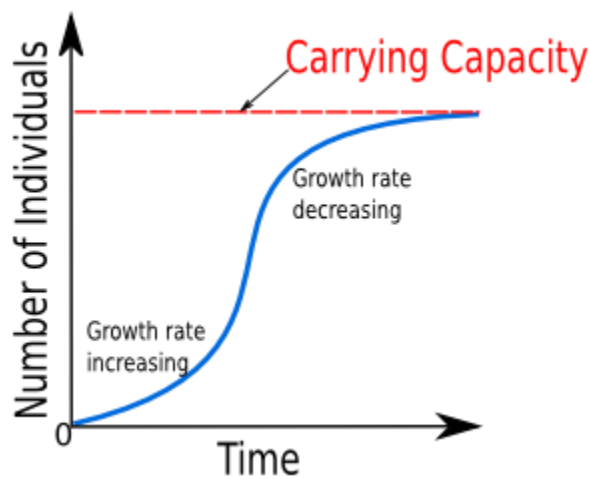
(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

Example 4.4.3: Bird Population

> The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1 + 25e^{-0.04t}}$$

Identify the initial population. What will be the bird population in five years? What will be the population in 150 years? What will be the population in 500 years?

We know the initial population, P_0, occurs when $t = 0$.

$$P_0 = P(0) = \frac{3640}{1 + 25e^{-0.04(0)}} = \frac{3640}{1 + 25e^{-0.04(0)}} = 140$$

Calculate the population in five years, when $t = 5$.

$$P(5) = \frac{3640}{1 + 25e^{-0.04(5)}} = \frac{3640}{1 + 25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years.

Calculate the population in 150 years, when $t = 150$.

$$P(150) = \frac{3640}{1 + 25e^{-0.04(150)}} = \frac{3640}{1 + 25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

Calculate the population in 500 years, when $t = 500$.

$$P(500) = \frac{3640}{1 + 25e^{-0.04(500)}} = \frac{3640}{1 + 25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds; but, slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.

Figure 4.4.5: Bird Population over a 200-Year Span

Bird Population — 3640

Time in Years — 500
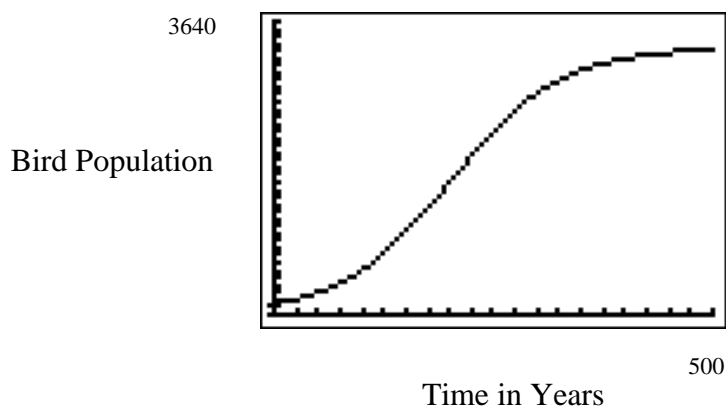
Example 4.4.4: Student Population at Northern Arizona University

The student population at NAU can be modeled by the logistic growth model below, with initial population taken from the early 1960's. We will use 1960 as the initial population date.

$$P(t) = \frac{30,000}{1 + 5e^{-0.06t}}$$

Determine the initial population and find the population of NAU in 2014. What will be NAU's population in 2050? From this model, what do you think is the carrying capacity of NAU?

We solve this problem by substituting in different values of time.

When $t = 0$, we get the initial population P_0.

$$P_0 = P(0) = \frac{30,000}{1 + 5e^{-0.06(0)}} = \frac{30,000}{6} = 5000$$

The initial population of NAU in 1960 was 5000 students.

In the year 2014, 54 years have elapsed so, $t = 54$.

$$P(54) = \frac{30,000}{1 + 5e^{-0.06(54)}} = \frac{30,000}{1 + 5e^{-3.24}} = \frac{30,000}{1.19582} = 25,087$$

There are 25,087 NAU students in 2014.

In 2050, 90 years has elapsed so, $t = 90$.

$$P(90) = \frac{30,000}{1+5e^{-0.06(90)}} = \frac{30,000}{1+5e^{-5.4}} = 29,337$$

There are 29,337 NAU students in 2050.

Finally, to predict the carrying capacity, look at the population 200 years from 1960, when $t = 200$.

$$P(200) = \frac{30,000}{1+5e^{-0.06(200)}} = \frac{30,000}{1+5e^{-12}} = \frac{30,000}{1.00003} = 29,999$$

Thus, the carrying capacity of NAU is 30,000 students.

It appears that the numerator of the logistic growth model, M, is the carrying capacity.

Carrying Capacity: Given the logistic growth model $P(t) = \dfrac{M}{1+ke^{-ct}}$, the carrying capacity of the population is M. M, the carrying capacity, is the maximum population possible within a certain habitat.

Example 4.4.5: Fish Population

Suppose that in a certain fish hatchery, the fish population is modeled by the logistic growth model where t is measured in years.

$$P(t) = \frac{12,000}{1+11e^{-0.2t}}$$

What is the carrying capacity of the fish hatchery? How long will it take for the population to reach 6000 fish?

The carrying capacity of the fish hatchery is $M = 12,000$ fish .

Now, we need to find the number of years it takes for the hatchery to reach a population of 6000 fish. We must solve for t when $P(t) = 6000$.

$$6000 = \frac{12,000}{1+11e^{-0.2t}}$$

$$\left(1+11e^{-0.2t}\right)\square 6000 = \frac{12,000}{1+11e^{-0.2t}}\square\left(1+11e^{-0.2t}\right)$$

$$\left(1+11e^{-0.2t}\right)6000 = 12,000$$

$$\frac{\left(1+11e^{-0.2t}\right)\cancel{6000}}{\cancel{6000}} = \frac{12,000}{6000}$$

$$1+11e^{-0.2t} = 2$$

$$11e^{-0.2t} = 1$$

$$e^{-0.2t} = \frac{1}{11} = 0.090909 \quad \text{Take the natural logarithm (ln on the calculator) of both}$$

sides of the equation.

$$\ln e^{-0.2t} = \ln 0.090909 \qquad \ln e^{-0.2t} = -0.2t \text{ by the rules of logarithms.}$$
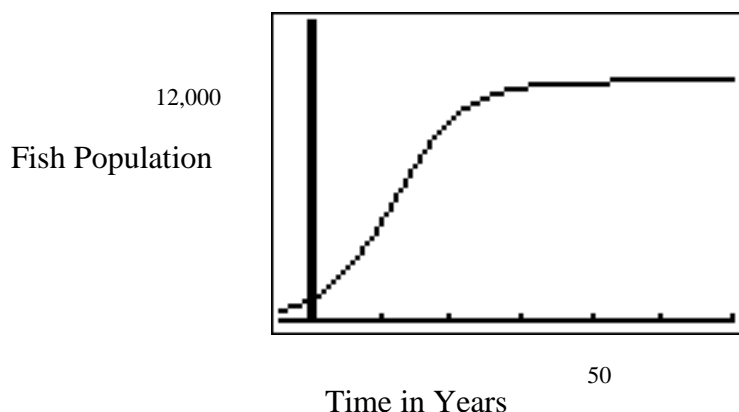
$$-0.2t = \ln 0.090909$$

$$t = \frac{\ln 0.090909}{-0.2}$$

$$t = 11.999$$

It will take approximately 12 years for the hatchery to reach 6000 fish.

Figure 4.4.6: Fish Population over a 30-Year Period.

12,000

Fish Population

Time in Years
50

Section 4.5: Modeling

We saw linear and exponential models used in previous sections of this chapter. In those sections, we were able to relate quantities through an equation (i.e. model) and solve problems that required finding a missing quantity. An extension of that concept is to use technology to create a model that closely fits an entire set of data values. Why would you want to do this? The model that is obtained can be used to understand the trends of the data set, make estimations of unknown data points, and explore the effects of changing a data

value. To aid in determining which of the many models to use to model the data, start with a graph called a scatter plot. The data is in ordered pairs of the independent variable, x, and the dependent variable, y. A scatter plot is just a plot of those ordered pairs. Then you look to see what type of model best fits the data. The models introduced previously in the chapter are linear, exponential, and logistic. There are many other models that exist, including but not limited to parabolic, cubic, and periodic.

Linear: $y = mx + b$

Exponential: $y = ab^x$ or alternatively $y = a(1+r)^t$
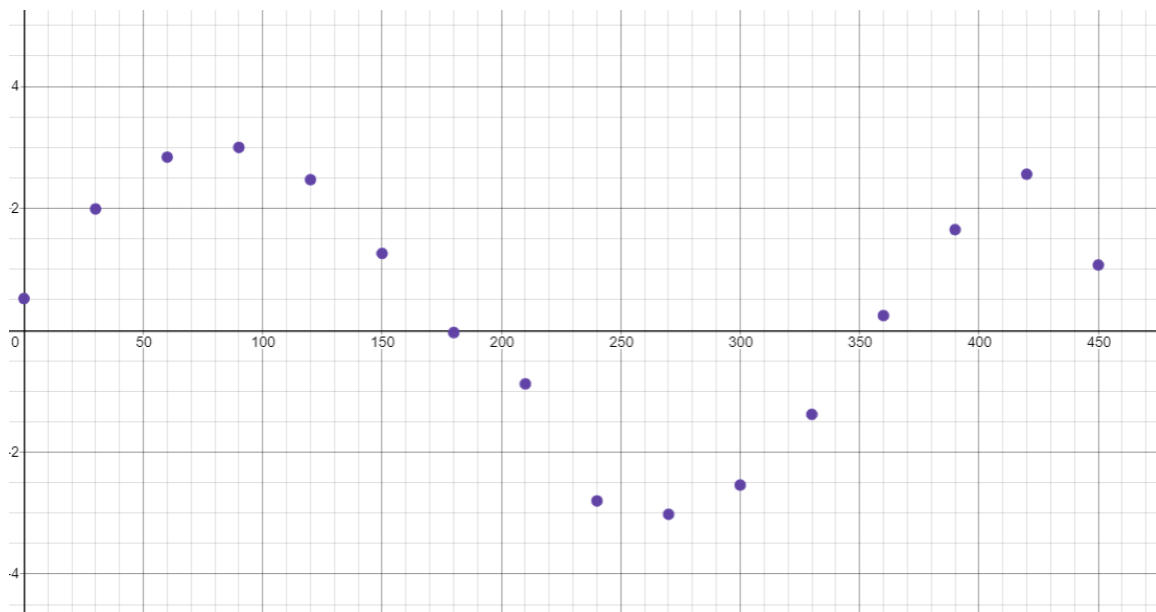
Logistic: $P(t) = \dfrac{M}{1 + ke^{-ct}}$

Parabolic: $y = ax^2 + bx + c$

Cubic: $y = ax^3 + bx^2 + cx + d$

Periodic: $y = a\sin(bx + c) + d$

Periodic models are ones that repeat after a certain time period. An example of a graph that would be periodic is:

Graph 4.5.1: Graph of Periodic Data

One way to determine if a model fits the data is to graph the model with the scatter plot and then look to see how well the model matches the data. Another way to assess if the model fits the data is a calculate the coefficient of determination, R^2. The coefficient of determination says what percentage of the variability of the dependent variable is explained by the model. The closer the coefficient of determination is to 100%, the stronger the model is in fitting the data.

Again, there are many other models that exist. These are just some of the more common models. Once you pick the model, you then must find the constants, called parameters, in the model. Technology can help to find those. The following example will demonstrate how to use Desmos to find the parameters. A later example will show how to find the parameters using the TI-84 calculator.

Example 4.5.1: Movie Ratings

There are several different movie rating sites. One question is do they compare with each other. Consider the following dataset. In it are the ratings for many movies from different sites. Use the dataset for Metacritc and Imdb to create a model to predict a rating for a movie on one system from the other system. (Hickey, 2015)

Table 4.5.2: Data Set of Movie Ratings

| Film	Year	Metacritic	Imdb
Wild Tales	2014	77	8.2
Testament of Youth	2015	77	7.3
The 100-Year-Old Man Who Climbed Out the Window and Disappeared	2015	58	7.1
Phoenix	2015	91	7.2
A Little Chaos	2015	51	6.4
Trainwreck	2015	75	6.7
Spy	2015	75	7.3
The Man From U.N.C.L.E.	2015	55	7.6
Tangerine	2015	86	7.4
Little Boy	2015	30	7.4
Tomorrowland	2015	60	6.6
Taken 3	2015	26	6.1
Mr. Turner	2014	94	6.9
Entourage	2015	38	7.1
Self/less	2015	34	6.6
Magic Mike XXL	2015	60	6.3
The Divergent Series: Insurgent	2015	42	6.4

Sinister 2	2015	31	5.5
Seventh Son	2015	30	5.5
Far From The Madding Crowd	2015	71	7.2
American Sniper	2015	72	7.4
Mission: Impossible â€" Rogue Nation	2015	75	7.8
Black Sea	2015	62	6.4
Still Alice	2015	72	7.5
Jurassic World	2015	59	7.3
The Vatican Tapes	2015	37	4.6
The Age of Adaline	2015	51	7.3
The Second Best Exotic Marigold Hotel	2015	51	6.6
Leviathan	2014	92	7.7
Chappie	2015	41	7
About Elly	2015	87	8.2
Ted 2	2015	48	6.6
Ex Machina	2015	78	7.7
Paul Blart: Mall Cop 2	2015	13	4.3
The DUFF	2015	56	6.6
Run All Night	2015	59	6.6
Danny Collins	2015	58	7.1
The Last Five Years	2015	60	6
Maggie	2015	52	5.6
Ricki and the Flash	2015	54	6.2
The Gallows	2015	30	4.4
Red Army	2015	82	7.7
Selma	2014	89	7.5
5 Flights Up	2015	55	6.1
Cinderella	2015	67	7.1
Paper Towns	2015	56	6.9
Monkey Kingdom	2015	72	7.3
Inside Out	2015	94	8.6
Welcome to Me	2015	67	5.9
The Diary of a Teenage Girl	2015	87	7

There are many types of technology that can be used to create this model. One type is Desmos (https://www.desmos.com/calculator). The process is first to copy the two columns of data for the Metacritc and Imdb ratings and paste them into Desmos. Desmos will automatically create a scatter plot of the data. To create the model, look at the data to determine which type of model best fits the data. Consider x to be the rating by Metacritic and y to be the rating by Imdb.

Figure 4.5.3: Desmos Scatterplot of Movie Ratings
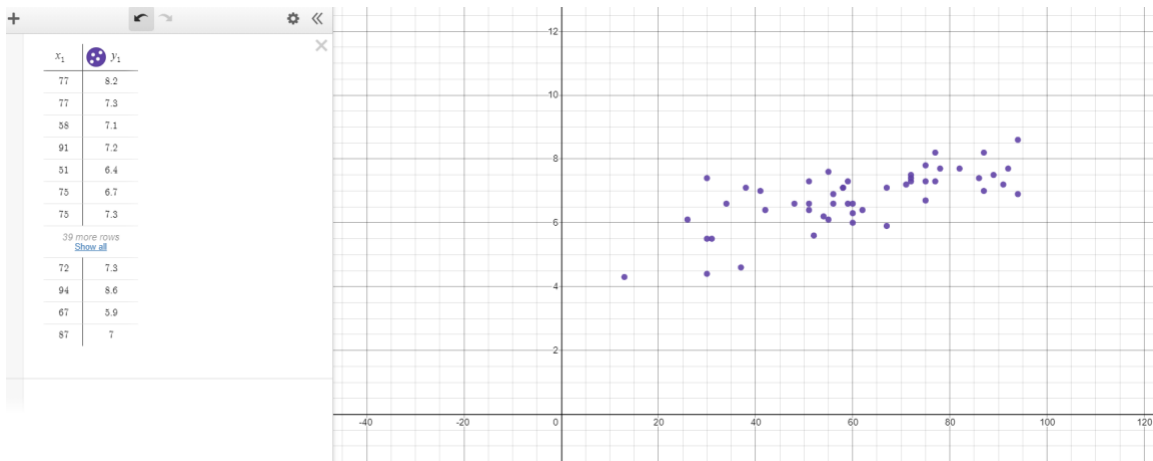
This data appears to have a positive linear relationship. So now you create a new input in Desmos and then type in the equation for the model that you think fits the data. In this case, since linear seems to fit the data, the equation for a linear model is $y = mx + b$. Type into Desmos, $y_1 \sim a * x_1 + b$. Notice when you type this into Desmos, the model line will be printed and it will calculate the parameter values, a and b.

Figure 4.5.4: Desmos Regression Line of Movie Ratings
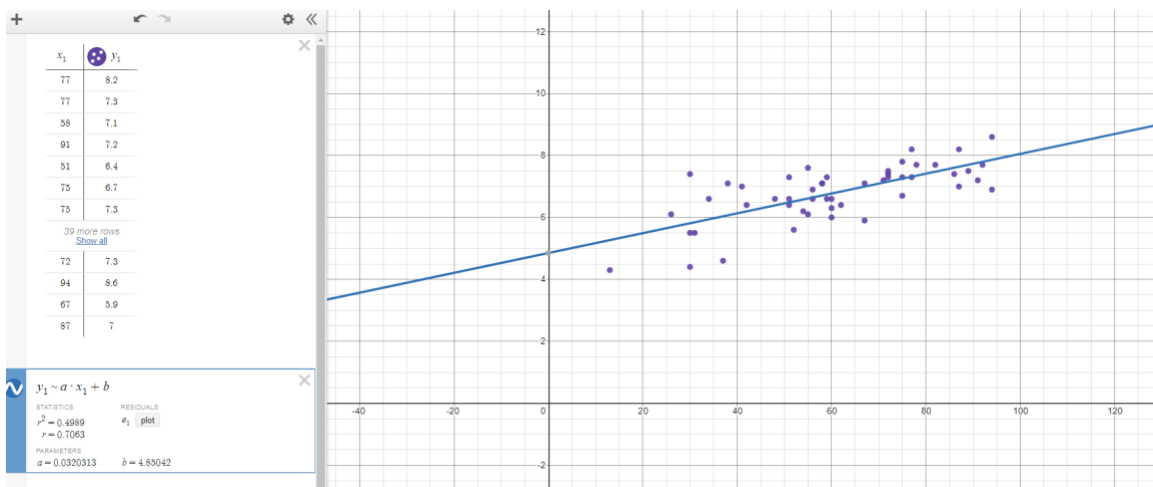
So $y = 0.032x + 4.85$ is the model describing the relationship between Metacritic and Imdb rating. Notice there is an $R^2 = 0.4989$. This means that 50% of the variability in Imdb rating can be explained by the model. This means this model does a fairly good job predicting the Imdb from the Metacritic rating. Now use the model to predict the Imdb rating of the Godzilla vs King Kong movie with a Metacritic score of 59.

_____

$$y = 0.032 * 59 + 4.856.3 = 6.738$$

By the way, the actual rating is 6.5, so the model does a pretty good job estimating the actual rating.

Example 4.5.2: M&M's® Experiment

Suppose there is an experiment where you take a snack size package of M&M's® and dump out all the M&M's® from the snack size package. Then take out all M&M's® that have the M showing. Count all the M&M's® that are left. Repeat this experiment with the remaining M&M's® repeatedly. Here is the data from someone who did this experiment several times.

Table 4.5.5: Data Set of M&M's®

Time period	M&M's®
0	50
1	27
2	18
3	10
4	6
5	6
6	4
7	3
8	3
9	3
10	3

Find the equation that models the number of M&M's® versus time.

This problem will use the TI-84. Go into the stat menu. Then go into 1: Edit… If you have data in the columns L1 and L2, clear that out, then type in the period data into L1 and the M&M's ® count into L2.

_____

Figure 4.5.6: Edit Window

Once you have the data typed in, you want to produce a scatter plot. To create this on the TI-84, you go into the statplot menu which is above the y= button (2nd y=). Go into 1: Plot_1. Set the window to look like this by arrowing to what you want to activate and pressing ENTER. You want the plot on, you want the Type to be the first plot, you want Xlist to be set to L1 (2nd 1), and the Ylist to be set to L2 (2nd 2). The mark and color is up to you, and if you have a different TI calculator you may not have the color option.

Figure 4.5.7: Stat Plot Window

Now you want to set the Window. The easiest way to do this is to go into the zoom menu and then pick 9:ZoomStat. Here is the graph:

Figure 4.5.8: Scatter Plot Window
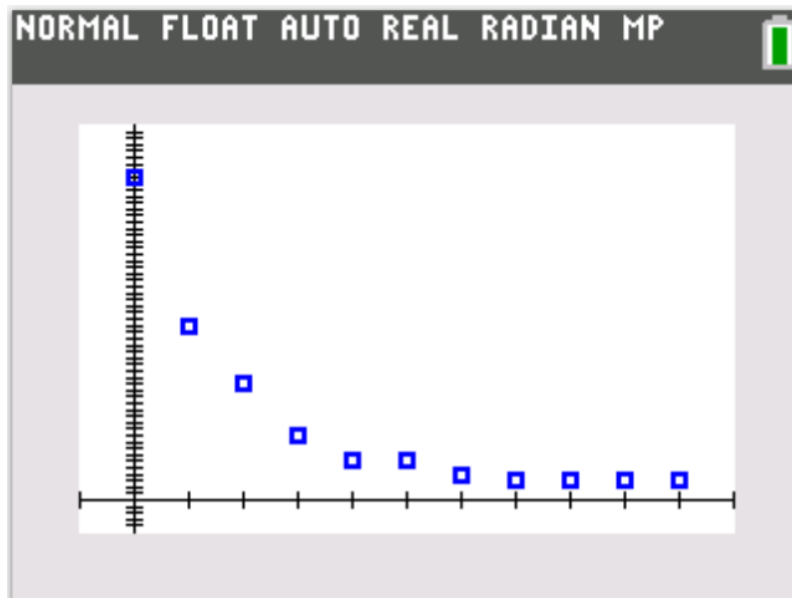
Looking at this scatter plot, it looks like a negative exponential function will model the data. Prior to finding the model, to find the coefficient of determination, R^2, you need to make sure the diagnostics are on. To do this press MENU.

Figure 4.5.9: Turn on R²
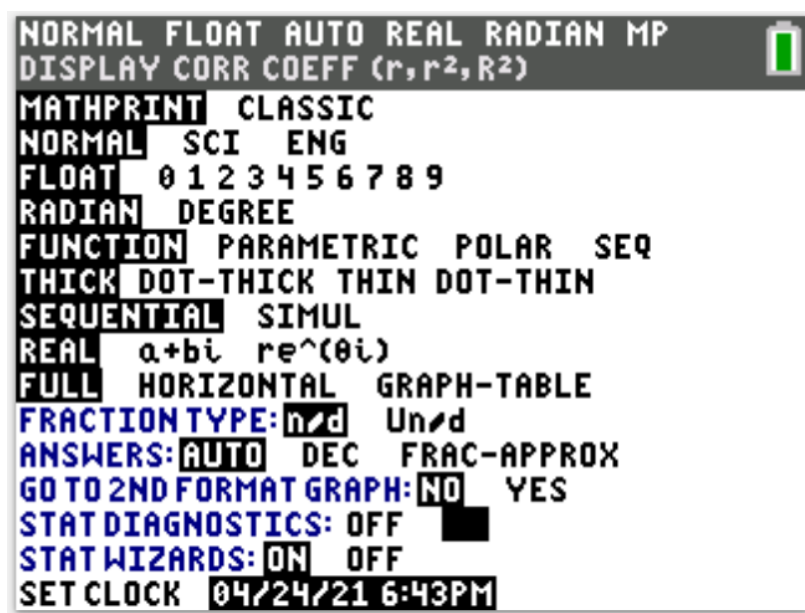
Then make sure that Stat DIAGNOSTICS is ON (move over to the word one and press ENTER).

To find the model with the TI-84, press stat menu and move over to CALC.

Figure 4.5.10: CALC Window
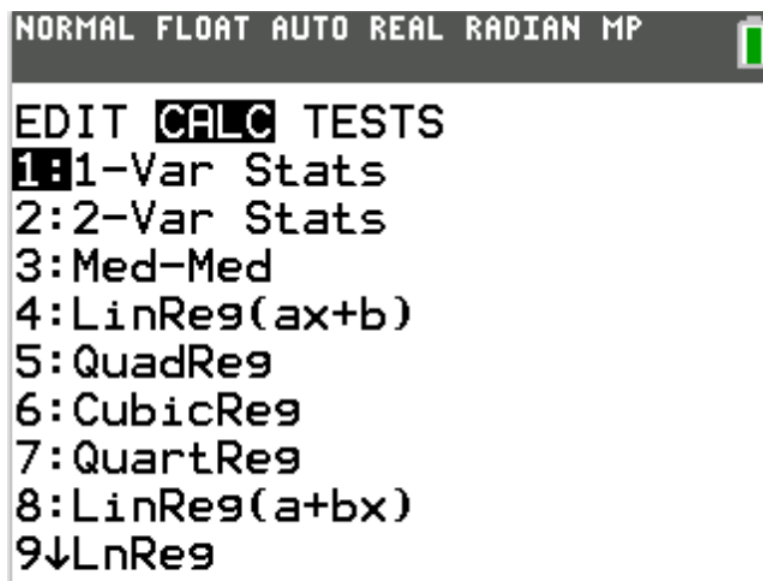
These are several different models that can be used for the data. The arrow next to number 9 means there are more models to choose from. Arrow down until you find 0:ExpReg. Go into this menu.

Figure 4.5.11: ExpReg Window
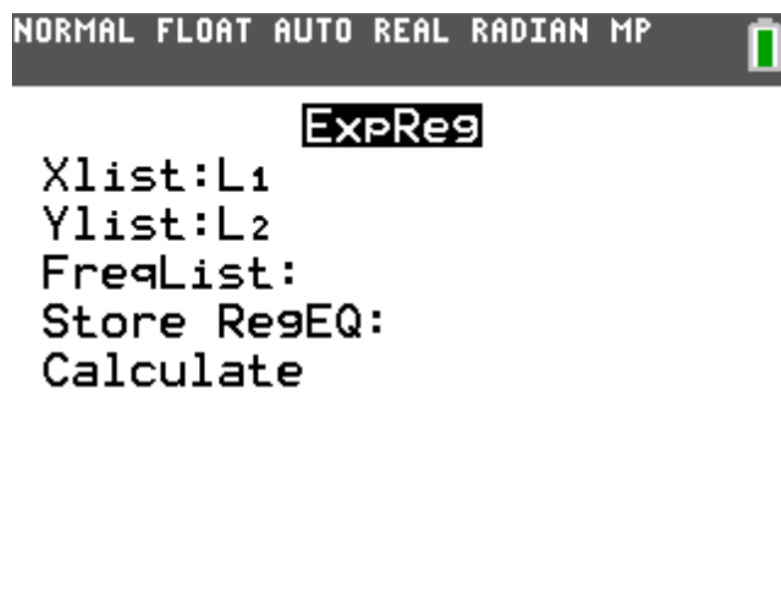
It should be set up for you, so just move down to Calculate and press enter.

Figure 4.5.12: ExpReg Window with R^2


```
NORMAL FLOAT AUTO REAL RADIAN MP

          ExpReg
 y=a*b^x
 a=29.6589403
 b=0.7541047392
 r²=0.8825359437
 r= ⁻0.9394338421
```
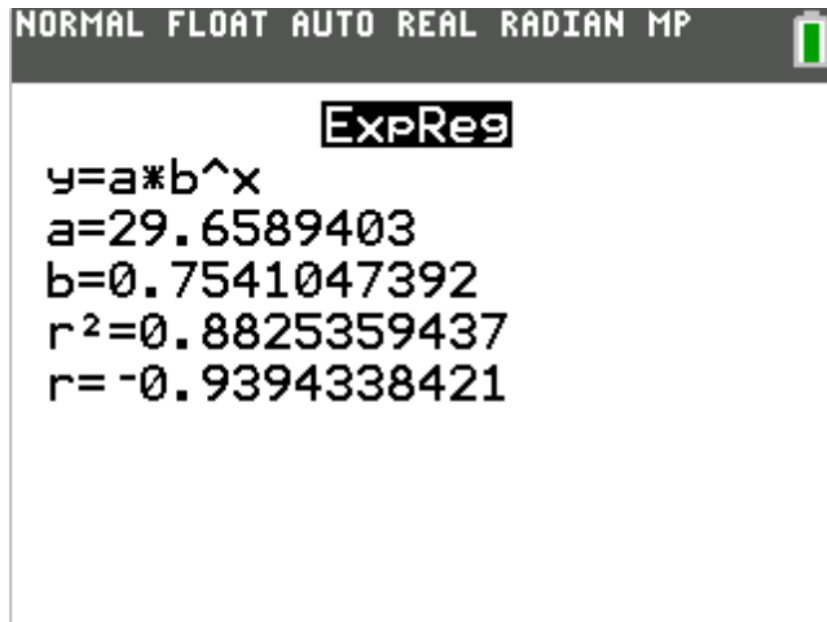
This is the equation where a and b are the parameters. So, the equation is

$$y = 29.7 * 0.75^x.$$

If you would like to graph the equation on the scatter plot, then you can go into y=, and either type in the equation in y1 or you can have the calculator put the equation in for you, by being in y= and then go into the 5:statistics menu and move over to EQ. Then press ENTER. This will put the last regression equation into y1. Now press graph. The model seems to model the data well. Notice that $r^2 = 0.88$. This means that 88% of the variability in number of M&M's® is explained by the period. This is a very strong relationship.

Figure 4.5.13: Exponential Regression Line
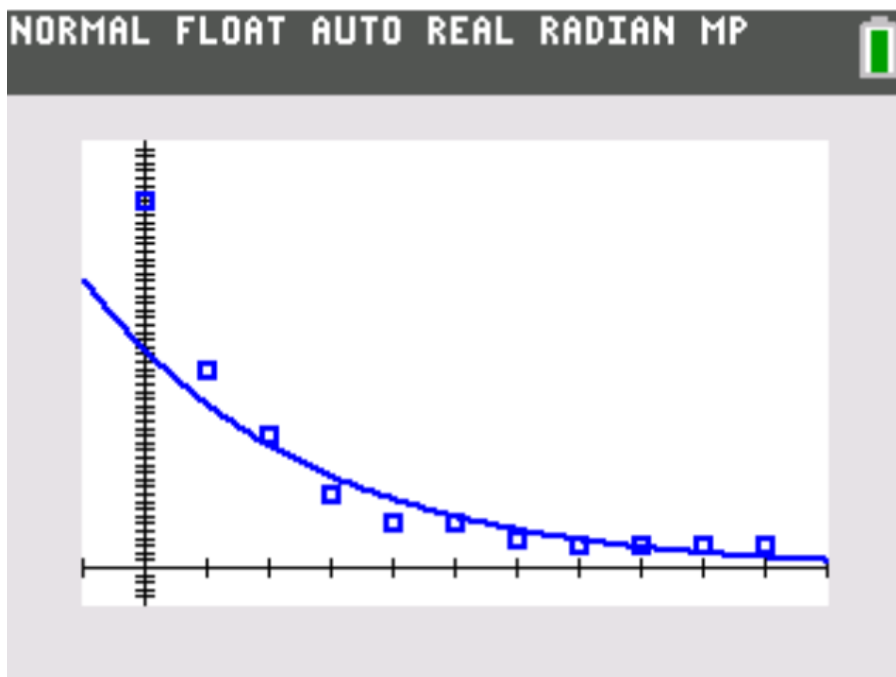
You can use the model to estimate how many M&M's® there are at the 12th time period. Input 12 into the model: $y = 29.7 * 0.75^{12} = 0.94$ or 1 M&M's.

Other datasets may produce other models. Use technology to create a scatter plot, and then use the graph to determine the model that would best fit the data. Then use technology to find the model. On Desmos you type in the general form of the model as shown in example 4.5.1. On the TI-84, pick the model from the list. Other technology can be used.

Chapter 4 Homework

1. Goodyear AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 72,800. The city's population grew by 3800 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Goodyear. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)

2. Gilbert AZ is one of the fastest growing cities in the nation according to the census bureau. In 2012, the population was about 229,800. The city's population grew by 9200 people from 2012 to 2013. If the growth keeps up in a linear fashion, create a population model for Gilbert. How many people will live there in 10 years? How many people will live there in 50 years? (U.S. Census, 2014)

3. Apache County AZ, is shrinking according to the census bureau. In 2012, the population was about 71,700. The county's population decreased by 1147 people from 2012 to 2013. If the decline is linear, create a population model for Apache County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

4. Cochise County AZ, is shrinking according to the census bureau. In 2012, the population was about 129,472. The county's population decreased by 2600 people from 2012 to 2013. If the decline is linear, create a population model for Cochise County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

5. Mohave County AZ, is shrinking according to the census bureau. In 2012, the population was about 203,030. The county's population decreased by 1200 people from 2012 to 2013. If the decline is linear, create a population model for Mohave County. How many people will live there in 10 years? How many people will live there in 50 years? (Kiersz, 2015)

6. The 2012 Kia Sedona LX has one of the largest depreciation values of any car. Suppose a 2012 Kia Sedona sold for $24,900, and its value depreciates by $3400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

7. The 2013 Chevy Impala has one of the largest depreciation values of any car. Suppose a 2013 Chevy Impala sold for $27,800 and its value depreciates by $3600 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

8. The 2013 Jaguar XJ AWD has one of the largest depreciation values of any car. Suppose a 2013 Jaguar XJ AWD sold for $74,500 and its value depreciates by $10,400 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

9. The 2012 Jeep Liberty Limited Sport 2WD has one of the largest depreciation values of any car. Suppose a 2012 Jeep Liberty Limited Sport 2WD sold for $23,400 and its value depreciates by $3,040 per year. Assuming the depreciation is linear, find a model for the depreciation. How much is the car worth in five years? How much is the car worth in 10 years? When is it worth nothing? (Fuscaldo, n.d.)

10. Suppose that in January, the maximum water depth at Lake Powell, Arizona was 528 feet. The water evaporates at an average rate of 1.2 feet per month. Find a model for the rate at which the water evaporates. If it does not rain at all, what will be the depth of Lake Powell in May and in September?

11. Suppose that the maximum water depth at Lake Tahoe, California in 2014 was 1644 feet. Because of the drought, the water level has been decreasing at an average rate of 6.2 feet per year. Find a model for the rate at which the water level decreases. If it there is no precipitation at all, what will be the depth of Lake Tahoe be in two years, and in five years?

12. Suppose the homes in Arizona have appreciated an average of 8% per year in the last five years. If the average home in a suburb sold for $225,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

13. Suppose the homes in Massachusetts have appreciated an average of 13% per year in the last five years. If the average home in a suburb sold for $205,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

14. Suppose the homes in Michigan have depreciated an average of 17% per year in the last five years. If the average home in a suburb sold for $215,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

15. Suppose the homes in Nevada have depreciated an average of 15% per year in the last five years. If the average home in a suburb sold for $318,000 in 2010, create a model for the home prices in the suburb. How much would this home be worth in 2015?

16. The cost of a home in Flagstaff AZ was $89,000 in 1992. In 2007, the same home appraised for $349,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?

17. The cost of a home in Bullhead City AZ was $109,000 in 1992. In 2007, the same home appraised for $352,000. Assuming the home value grew according to the exponential growth model, find the annual growth rate of this home over this 15-year period. If the growth continued at this rate, what would the home be worth in 2020?

18. The population of West Virginia is in decline. The population in 2014 was 1,850,326 and the population had decreased by 0.14% from 2010. How many people were living in West Virginia in 2010? Create a model for this population. If the decline continues at this rate, how many people will reside in West Virginia in 2020? (Wikipedia, n.d.)

19. Assume that the population of Arizona grew by 2.4% per year between the years 2000 to 2010. The number of Native American living in Arizona was 257,426 in 2010. How many Native Americans were living in Arizona in 2000? Create a model for this population. If the growth continues at this rate, how many Native Americans will reside in Arizona in 2020?

20. Assume the population of the U.S. grew by 0.96% per year between the years 2000 and 2010. The number of Hispanic Americans was 55,740,000 in 2010. How many Hispanic Americans were living in the U.S. in 2000? Create a model for this population. If the growth continues at this rate, how many Hispanic Americans will reside in The U.S. in 2020?

21. Assume the population of Michigan decreased by 0.6% per year between the years 2000 to 2010. The population of Michigan was 9,970,000 in 2010. How many people were living in Michigan in 2000? Create a model for this population. If the growth continues at this rate, how many people will reside in Michigan in 2020?

22. The doubling time of a population of aphids is 12 days. If there are initially 200 aphids, how many aphids will there be in 17 days?

23. The doubling time of a population of rabbits is six months. If there are initially 26 rabbits, how many rabbits will there be is 17 months?

24. The doubling time of a population of shrews is three months. If there are initially 32 shrews, how many shrews will there be is 21 months?

25. The doubling time of a population of hamsters is 1.2 years. If there are initially 43 hamsters, how many hamsters will there be is 7 years?

26. A certain cancerous tumor doubles in size in three months. If the initial size of the tumor is two cells, how many months will it take for the tumor to grow to 60,000 cells? How many cells will there be in 1.5 years? In three years?

27. A certain cancerous tumor doubles in size in 1.5 months. If the initial size of the tumor is eight cells, how many months will it take for the tumor to grow to 40,000 cells? How many cells will there be in six months? In 2.5 years?

28. A bird population on a certain island has an annual growth rate of 1.5% per year. Approximate the number of years it will take the population to double. If the initial population is 130 birds, use it to find the bird population of the island in 14 years.

29. The beaver population on Kodiak Island has an annual growth rate of 1.2% per year. Approximate the number of years it will take the population to double. If the initial population is 32 beavers, use it to find the population of beavers on the island in 20 years.

30. The black-footed ferret population in Arizona has an annual growth rate of 0.5% per year. Approximate the number of years it will take the population to double. If the initial population is 230 ferrets, use it to find the ferret population in AZ in 12 years.

31. The Mexican gray wolf population in southern Arizona increased from 72 individuals to 92 individuals from 2012 to 2013. What is the annual growth rate? Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Mexican gray wolves in 10 years.

32. There is a small population of Sonoran pronghorn antelope in a captive breeding program on the Cabeza Prieta National Wildlife Refuge in southwest Arizona. Recently, the population increased from 122 individuals to 135 individuals from 2012 to 2013. Approximate the number of years it will take the population to double. Create the doubling time model and use it to find the population of Sonoran pronghorn antelope in 10 years.

33. Lead-209 is a radioactive isotope. It has a half-life of 3.3 hours. Suppose that 56 milligrams of this isotope is created in an experiment, how much is left after 12 hours?

34. Titanium-44 has a half-life of 63 years. If there is 560 grams of this isotope, how much is left after 1200 years?

35. Uranium-232 has a half-life of 68.9 years. If there is 160 grams of this isotope, how much is left after 1000 years?

36. Nickel-63 has a half-life of 100 years. If there is 16 grams of this isotope, how much is left after 145 years?

37. Radium-226 has a half-life of 1600 years. If there is 56 grams of this isotope, how much is left after 145,000 years?

38. The population of wild desert tortoises is decreasing by 72% per year. Approximate the half-life for this population. If there are currently 100,000 tortoises left in the wild, how many will remain in 20 years?

39. The population of pygmy owls is decreasing by 4.4% per year. Approximate the half-life for this population. If there are currently 41 owls left in the wild, how many will remain in five years?

40. Radioactive carbon-14 is used to determine the age of artifacts because it concentrates in organisms only when they are alive. It has a half-life of 5730 years. In 2004, carbonized plant remains were found where human artifacts were unearthed along the Savannah River in Allendale County. Analysis indicated that the plant remains contained 0.2% of their original carbon-14. Estimate the age of the plant remains. (Wikipedia, n.d.)

41. The population of a bird species on an island grows according to the logistic model below.

$$P(t) = \frac{43,240}{1+12e^{-0.05t}}$$

Identify the initial population. What will be the bird population in five years? In 150 years? In 500 years? What is the carrying capacity of the bird population on the island?

42. Suppose the population of students at Arizona State University grows according to the logistic model below with the initial population taken from 2009.

$$P(t) = \frac{85,240}{1+0.6e^{-0.15t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at ASU?

43. Suppose the population of students at Ohio State University grows according to the logistic model below with the initial population taken from 1970.

$$P(t) = \frac{90,000}{1+4e^{-0.06t}}$$

Identify the initial population. What will be the campus population in five years? What will be the population in 20 years? What is the carrying capacity of students at OSU?

44. Suppose that in a certain shrimp farm, the shrimp population is modeled by the logistic model below where t is measured in years.

$$P(t) = \frac{120,000}{1+25e^{-0.12t}}$$

Find the initial population. How long will it take for the population to reach 6000 shrimp? What is the carrying capacity of shrimp on the farm?

45. Suppose that in a certain oyster farm, the oyster population is modeled by the logistic model below where t is measured in years.

$$P(t) = \frac{18,000}{1+19e^{-0.36t}}$$

Find the initial population. How long will it take for the population to reach 2000 oysters? What is the carrying capacity of oysters on the farm?

46. Nutrition can affect the height of a person. Consider the number of calories from animal protein eaten by a man in a country and the average height of a male in that country are in the following table. (Roser, M. et al., n.d.)

Country	year	calories	height
Afghanistan	1969	68.76	165.1135
Algeria	1992	74.12	170.6454
Argentina	1964	226.84	172.3648
Austria	1990	263.08	177.5834
Azerbaijan	1978	205.6	171.8626
Bahamas	1985	197.56	172.56
Barbados	1989	248.44	175.8113
Belize	1963	91.48	168.1696
Botswana	1987	78.56	172.6642
Burkina Faso	1971	26.2	171.3792
Canada	1982	237.12	177.5477
Chile	1966	96.4	169.6918
China	1969	22.88	168.6978
Congo	1972	55.88	170.4727
Congo	1969	57.16	170.5026
Cuba	1981	144.4	171.3655
Djibouti	1990	42.6	167.5047
Dominica	1965	89.48	174.6352
Gabon	1995	160.28	168.132
Grenada	1966	100.32	175.3355
Guinea-Bissau	1988	34.24	169.3965
Iraq	1970	52.12	170.8853
Kiribati	1981	109.44	169.9326
Lesotho	1990	39.84	166.6611
Malaysia	1973	73.2	166.7981

Malaysia	1994	167.08	167.8264
Maldives	1975	130.44	164.6324
Moldova	1980	204.16	174.7058
New Zealand	1967	246.92	177.1598
Norway	1963	204.64	179.3945
Peru	1987	92.32	165.2174
Peru	1965	79.8	164.3932
Poland	1974	238.76	176.6568
Saint Vincent and the Grenadines	1994	125.16	172.7317
Saudi Arabia	1966	35.8	168.1402
Slovenia	1993	200.08	179.7296
South Africa	1962	93.32	169.2712
South Africa	1990	106.44	167.7651
Spain	1982	230.12	175.8251
Sri Lanka	1977	29.76	164.697
Suriname	1978	110.84	172.4756
Tanzania	1979	41.6	167.2221
Thailand	1976	61.48	166.2704
Timor	1976	103.96	161.0374
Togo	1961	20.16	169.908
Turkey	1970	100.12	172.402
Turkey	1969	102.36	172.2595
Uganda	1962	43.32	169.5561
Uruguay	1970	249.64	172.9225
Venezuela	1986	115.4	171.2487

a. Use technology to create a scatterplot for the data.
b. Use technology to create an appropriate model for the mean height of a male versus the number of calories from animal protein eaten by a person.
c. Determine the R^2 value of your model and interpret it.
d. Use your model to estimate the mean height of a male that eats 50 kcal of calories from animal protein.
e. Should your estimate value be trusted? Why or why not?
f. Use your model to the mean height of a male that eats 300 kcal of calories from animal protein. Should your predicted value be trusted? Why or why not?

47. The data for the number of cases of COVID-19 in the United States from the start of 2020. (Roser, M. et al., n.d.)

Month	Total cases
0	1
1	16
2	34898
3	854288
4	1611249
5	2313093
6	3974341
7	5664641
8	6899723
9	8429792
10	12347905
11	18321157
12	24846933
13	28163046
14	29096048

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the number of cases based on the month.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the number of cases of COVID-19 in the United States 18 months after the start of 2020.
    e. Should your estimate value be trusted? Why or why not?
    f. Use your model to predict number of cases of COVID-19 in the United States 24 months after the start of 2020. Should your predicted value be trusted? Why or why not?

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48. The amount of sunlight that a location has per day is of interest in the use of solar power for creating electricity. The dataset has the amount of sunlight hours in Finland in a year. (Roser, M. et al., n.d.)

date	day	Length (hours)
1-Jan	1.00	6
1-Feb	31.00	7.95
1-Mar	59.00	10.43333
1-Apr	90.00	13.28333
1-May	120.00	16
1-Jun	151.00	18.35
1-Jul	181.00	18.78333
1-Aug	212.00	16.88333
1-Sep	243.00	14.18333
1-Oct	273.00	11.48333
1-Nov	304.00	7.45
1-Dec	334.00	6.433333
31-Dec	365.00	5.95

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the length of daylight hours versus day.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the length of daylight hours 160 days into the year.
    e. Should your estimate value be trusted? Why or why not?

49. When pouring a carbonated beverage, foam appears on top of the liquid and gradually decreases. Below is a table the height of the foam after pouring. (Hackbarth, 2006)

Time from Pour (seconds)	Foam Height (cm)
0	17.4
15	15.1
30	13.1
45	11.6
60	10.6
90	8.7
120	7.4
150	6.35
180	5.4
210	4.5

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240	3.8
270	3.3
300	2.9

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the height of the foam after 100 seconds. Should your estimate value be trusted? Why or why not?
    e. Use your model to predict the height of the foam after 350 seconds. Should your predicted value be trusted? Why or why not?

50. The table below shows the S & P 500 stock index values for the years 1930 through 2020 in 5-year increments. (USA Facts, n.d.)

Year	S&P 500 values
1930	15.34
1935	13.43
1940	10.58
1945	17.36
1950	20.43
1955	45.48
1960	58.11
1965	92.43
1970	92.15
1975	90.19
1980	135.76
1985	211.28
1990	330.22
1995	615.93
2000	1320.28
2005	1248.29
2010	1257.64
2015	2043.94
2020	3756.07

    a. Use technology to create a scatterplot of the data set.
    b. Use technology to create an appropriate model of the data set.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to predict the S & P stock index value in 2002. Should your predicted value be trusted? Why or why not?
    e. Use your model to predict the S & P stock index value in 2030. Should your predicted value be trusted? Why or why not?

51. The data below lists information about 50 buildings: the year in which it was built, the height in feet, and the number of stories. (Penn State, n.d.a)

Year	Height in feet	Number of Stories
1990	770	54
1980	677	47
1990	428	28
1989	410	38
1966	371	29
1976	504	38
1974	1136	80
1991	695	52
1982	551	45
1986	550	40
1931	568	49
1979	504	33
1988	560	50
1973	512	40
1981	448	31
1983	538	40
1968	410	27
1927	409	31
1969	504	35
1988	777	57
1987	496	31
1960	386	26
1984	530	39
1976	360	25
1920	355	23
1931	1250	102
1989	802	72
1907	741	57
1988	739	54
1990	650	56
1973	592	45
1983	577	42
1971	500	36
1969	469	30
1971	320	22
1988	441	31
1989	845	52
1973	435	29
1987	435	34
1931	375	20
1931	364	33

1924	340	18
1931	375	23
1991	450	30
1973	529	38
1976	412	31
1990	722	62
1983	574	48
1984	498	29
1986	493	40

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to predict the height of a 41-story building. Should your predicted value be trusted? Why or why not?

52. The table below shows the number of cricket chirps observed per fifteen seconds and the outside temperature at the time of observation. (Peggy, 2007)

Cricket Chirps per 15 seconds	Temperature (°F)
44	80.5
35	70.5
20.4	57
33	66
31	68
35	72
18.5	52
37	73.5
26	53

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the number of cricket chirps per fifteen seconds if the temperature were 77°F. Should your estimate value be trusted? Why or why not?
    e. Use your model to predict the number of cricket chirps per fifteen seconds if the temperature were 32°F. Should your predicted value be trusted? Why or why not?

53. The data set below gives the one repetition maximum power clean (weightlifting) weight in kilograms and the personal best shot put in meters for 28 female collegiate shot putters. (USA Facts, n.d.)

Power clean (kg)	Shot put (meters)
37.5	6.4
51.5	10.2
61.3	12.4
61.3	13
63.6	13.2
66.1	13
70	12.7
92.7	13.9
90.5	15.5
90.5	15.8
94.8	15.8
97	16.8
97	17.1
97	17.8
102	14.8
102	15.5
103.6	16.1
100.4	16.2
108.4	17.9
114	15.9
115.3	15.8
114.9	16.7
114.7	17.6
123.6	16.8
125.8	17
119.1	18.2
118.9	19.2
141.1	18.6

a. Use technology to create a scatterplot of the data set.
b. Use technology to create an appropriate model for data set.
c. Determine the R^2 value of your model and interpret it.
d. Maggie Ewen, representing Arizona State, set the collegiate record for the women's shot put in 2018. Her record setting shot put distance was 19.46 meters. Use your model from part b to predict Maggie's power clean weight.

54. The table below shows the weight in pounds for a dog at different amounts of days since its birth. (USA Facts, n.d.)

Days Since Birth	Mass (lbs)
31	6.25
62	10
93	20
99	23
107	26
113	27.6
121	29.8
127	31.6
148	37.2
161	41.2
180	48.7
214	54
221	54
307	63
452	66
482	72
923	72.2
955	76
1308	75

    a. Use technology to create a scatterplot of the data set.
    b. Use technology to create an appropriate model for the data set.
    c. Determine the R^2 value of your model and interpret it.
    d. Use the model to predict the weight of the dog at 2000 days since birth. Explain the value that you obtained.

55. Gross domestic product (GDP) is the total value of goods produced and services provided in a country during one year. Below is a table showing the United States GDP in millions of dollars for every other year from 1929 to 2017. (USA Facts, 2020)

Year	US GDP in Millions of Dollars
1929	104556
1931	77391
1933	57154
1935	74241
1937	93003
1939	93437

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1941	129309
1943	203084
1945	228007
1947	249616
1949	272475
1951	346914
1953	389218
1955	425478
1957	474039
1959	521654
1961	562210
1963	637451
1965	742289
1967	859958
1969	1017615
1971	1164850
1973	1425376
1975	1684904
1977	2081826
1979	2627334
1981	3207042
1983	3634038
1985	4338979
1987	4855215
1989	5641580
1991	6158129
1993	6858559
1995	7639749
1997	8577552
1999	9630663
2001	10581822
2003	11458246
2005	13036637
2007	14451860
2009	14448932
2011	15542582
2013	16784851
2015	18238301
2017	19542980

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the US GDP in 1990. Should your estimated value be trusted? Why or why not?
    e. Use your model to predict the US GDP in 2050. Should your predicted value be trusted? Why or why not?

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56. Below is the population of the United States as recorded by the United
    Stated Census Bureau every ten years from 1790 to 2010. (Penn State, n.d.b)

1790	3.929
1800	5.308
1810	7.24
1820	9.638
1830	12.866
1840	17.069
1850	23.192
1860	31.443
1870	38.558
1880	50.156
1890	62.948
1900	76.212
1910	92.228
1920	106.021
1930	123.202
1940	132.164
1950	151.325
1960	179.323
1970	203.302
1980	226.542
1990	248.71
2000	281.421
2010	308.746

   a. Use technology to create a scatterplot for the data.
   b. Use technology to create an appropriate model for the data.
   c. Determine the R^2 value of your model and interpret it.
   c. Use your model to estimate the US population in 1987. Should
      your estimated value be trusted? Why or why not?
   d. Use your model to predict the US population in 2050. Should your
      predicted value be trusted? Why or why not?

57. Below is a table showing the number of children served by the foster care system
    in the United States each year from 2010 to 2019. (USA Facts, 2019b)

Year	Children in Foster Care
2010	663000
2011	638000

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2012	630000
2013	633000
2014	646000
2015	663406
2016	680433
2017	685007
2018	686151
2019	672594

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the number of children served by the foster care system in 2020. Should your predicted value be trusted? Why or why not?
    e. Do you think it would be reasonable to use this model to predict the number of children in foster care in 2050? Why or why not?

58. As the Moon orbits Earth, the amount of sunlight cast on the Moon changes, which affects how much of the Moon we can see from Earth. The phase of the Moon can be expressed as a percent of the face of the Moon visible from Earth. For example, when there is a full moon we can see 100% of the Moon's face; when there's a half-moon, we can see 50% of the Moon's face, and so on. Below is a table of the phase of the Moon at midnight every night from January 1, 2020, to February 29, 2020. (NASA, 2019)

Days after Jan 1, 2020	Phase
0	29.95
1	38.92
2	48.26
3	57.72
4	67.03
5	75.88
6	83.93
7	90.78
8	96
9	99.18
10	99.94
11	98.08
12	93.58
13	86.67
14	77.76
15	67.41
16	56.26

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17	44.95
18	34.05
19	24.1
20	15.53
21	8.65
22	3.72
23	0.85
24	0.05
25	1.27
26	4.34
27	9.07
28	15.23
29	22.59
30	30.9
31	39.94
32	49.45
33	59.17
34	68.79
35	77.92
36	86.11
37	92.84
38	97.56
39	99.78
40	99.19
41	95.71
42	89.55
43	81.19
44	71.25
45	60.42
46	49.34
47	38.57
48	28.59
49	19.75
50	12.34
51	6.59
52	2.61
53	0.51
54	0.27
55	1.85
56	5.16
57	10.05
58	16.37
59	23.92

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      a. Use technology to create a scatterplot for the data.
      b. Use technology to create an appropriate model for the data.
      c. Determine the R^2 value of your model and interpret it.
      d. Use your model to predict the moon phase on April 1, 2020. Should your predicted value be trusted? Why or why not?

59. Many Americans have been affected by the opioid epidemic over the last few decades, and you have probably heard about it in the news. The table below shows the number of opioid prescriptions administered per 100 people in the United States in each year from 2006 to 2019. (USA Facts, 2019a)

Year	Opioid Dispensing Rate per 100 people
2006	72.4
2007	75.9
2008	78.2
2009	79.5
2010	81.2
2011	80.9
2012	81.3
2013	78.1
2014	75.6
2015	70.6
2016	66.5
2017	59
2018	51.4
2019	46.7

      a. Use technology to create a scatterplot for the data.
      b. Use technology to create an appropriate model for the data.
      c. Determine the R^2 value of your model and interpret it.
      d. Use your model to predict the number of opioid prescriptions administered in the US per 100 people in 2020. Should your predicted value be trusted? Why or why not?
      e. Use your model to predict the number of opioid prescriptions administered in the US per 100 people in 2040. Should your predicted value be trusted? Why or why not?

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60. Below is a table showing the results from a study on the relationship between driving speed and fuel use in cars. (Franklin, 2006)

Speed (miles per hour)	Mileage (miles per gallon)
15	22.3
20	25.5
25	27.5
30	29
35	28.8
40	30
45	29.9
50	30.2
55	30.4
60	28.8
65	27.4
70	25.3
75	23.3

    a. Use technology to create a scatterplot for the data.
    b. Use technology to create an appropriate model for the data.
    c. Determine the R^2 value of your model and interpret it.
    d. Use your model to estimate the mileage at a speed of 62 miles per hour. Should your predicted value be trusted? Why or why not?

61. The motion for a ball being thrown up is in the table.

time (s)	height (cm)
0	-0.23363
0.0333	7.008868
0.0667	12.84959
0.1	17.98943
0.133	22.19475
0.167	24.76467
0.2	26.6337
0.233333	27.33458
0.266667	27.10095
0.3	25.46555
0.333333	22.66201
0.366667	18.92394
0.4	14.25136
0.433333	8.177012
0.466667	0.934516

a. Use technology to create a scatterplot for the data.
b. Use technology to create an appropriate model for the data.
c. Determine the R^2 value of your model and interpret it.
d. Use your model to estimate the height at 0.25 seconds. Should your predicted value be trusted? Why or why not?
e. Use your model to predict the height at 0.6 seconds. Should your predicted value be trusted? Why or why not?