Chapter 9: Apportionment

Apportionment involves dividing something up, just like fair division. In fair division we are dividing objects among people while in apportionment we are dividing people among places. Also like fair division, the apportionment processes that are widely used do not always give the best answer, and apportionment is still an open field of mathematics.

Apportionment is used every day in American politics. It is used to determine the size of voting districts and to determine the number of representatives from each state in the U.S. House of Representatives. Another example of how apportionment can be used is to assign a group of new fire fighters to the fire stations in town in an equitable way. Overall, apportionment is used to divide up resources (human or otherwise) in as fair a way as possible.

Section 9.1 Basic Concepts of Apportionment and Hamilton's Method

Apportionment can be thought of as dividing a group of people (or other resources) and assigning them to different places.

Example 9.1.1: Why We Need Apportionment

> Tom is moving to a new apartment. On moving day, four of his friends come to help and stay until the job is done since Tom promised they will split a case of beer afterwards. It sounds like a fairly simple job to split the case of beer between the five friends until Tom realizes that 24 is not evenly divisible by five. He could start by giving each of them (including himself) four beers. The question is how to divide the four remaining beers among the five friends assuming they only get whole beers. Apportionment methods can help Tom come up with an equitable solution

Basic Concepts of Apportionment:

The apportionment methods we will look at in this chapter were all created as a way to divide the seats in the U.S. House of Representatives among the states based on the size of the population for each state. The terminology we use in apportionment reflects this history. An important concept is that the number of seats a state has is proportional to the population of the state. In other words, states with large populations get lots of seats and states with small populations only get a few seats.

The **seats** are the people or items that are to be shared equally. The **states** are the parties that will receive a proportional share of the seats.

The first step in any apportionment problem is to calculate the standard divisor. This is the ratio of the total population to the number of seats. It tells us how many people are represented by each seat.

The **standard divisor** is $SD = \dfrac{\text{total population}}{\text{\# seats}}$.

The next step is to find the standard quota for each state. This is the exact number of seats that should be allocated to each state if decimal values were possible.

The **standard quota** is $SQ = \dfrac{\text{state population}}{\text{standard divisor}}$

Example 9.1.2: Finding the Standard Quota

Hamiltonia, a small country consisting of six states is governed by a senate with 25 members. The number of senators for each state is proportional to the population of the state. The following table shows the population of each state as of the last census.

Table 9.1.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

Find the standard divisor and the standard quotas for each of the states of Hamiltonia.

Standard Divisor: $SD = \dfrac{\text{total population}}{\text{\# seats}} = \dfrac{237,000}{25} = 9480$

This means that each seat in the senate corresponds to a population of 9480 people.

Standard Quotas:

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Alpha: $SQ = \dfrac{\text{state population}}{\text{standard divisor}} = \dfrac{24{,}000}{9480} = 2.532$

Beta: $SQ = \dfrac{\text{state population}}{\text{standard divisor}} = \dfrac{56{,}000}{9480} = 5.907$

If fractional seats were possible, Alpha would get 2.532 seats and Beta would get 5.907 seats.

Use similar calculations for the other states.

Table 9.1.2: Standard Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001

Notice that the sum of the standard quotas is 25.001, the total number of seats. This is a good way to check your arithmetic.

Note: Do not worry about the 0.001. That is due to rounding and is negligible.

The standard quota for each state is usually a decimal number but in real life the number of seats allocated to each state must be a whole number. Rounding off the standard quota by the usual method of rounding does not always work. Sometimes the total number of seats allocated is too high and other times it is too low. In Example 9.1.2 the total number of seats allocated would be 26 if we used the usual rounding rule.

When we round off the standard quota for a state the result should be the whole number just below the standard quota or the whole number just above the standard quota. These values are called the lower and upper quotas, respectively. In the extremely rare case that the standard quota is a whole number, use the standard quota for the lower quota and the next higher integer for the upper quota.

The **lower quota** is the standard quota rounded down. The **upper quota** is the standard quota rounded up.

Example 9.1.3: Upper and Lower Quotas for Hamiltonia

Find the lower and upper quotas for each of the states in Hamiltonia.

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Table 9.1.3: Upper and Lower Quotas for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20
Upper Quota	3	6	3	2	7	5	26

Note: The total of the lower quotas is 20 (below the number of seats to be allocated) and the total of the upper quotas is 26 (above the number of seats to be allocated).

Hamilton's Method

The U.S. Constitution requires that the seats for the House of Representatives be apportioned among the states every ten years based on the sizes of the populations. Since 1792, five different apportionment methods have been proposed and four of these methods have been used to apportion the seats in the House of Representatives. The number of seats in the House has also changed many times. In many situations the five methods give the same results. However, in some situations, the results depend on the method used. As we will see in the next section, each of the methods has at least one weakness. Because it was important for a state to have as many representatives as possible, senators tended to pick the method that would give their state the most representatives. In 1941, the number of seats in the House was fixed at 435 and an official method was chosen. This took the politics out of apportionment and made it a purely mathematical process.

Alexander Hamilton proposed the first apportionment method to be approved by Congress. Unfortunately for Hamilton, President Washington vetoed its selection. This veto was the first presidential veto utilized in the new U.S. government. A different method proposed by Thomas Jefferson was used instead for the next 50 years. Later, Hamilton's method was used off and on between 1852 and 1901.

Summary of Hamilton's Method:

1. Use the standard divisor to find the standard quota for each state.
2. Temporarily allocate to each state its lower quota of seats. At this point, there should be some seats that were not allocated.
3. Starting with the state that has the largest fractional part and working toward the state with the smallest fractional part, allocate one additional seat to each state until all the seats have been allocated.

Example 9.1.4: Hamilton's Method for Hamiltonia

Use Hamilton's method to finish the allocation of seats in Hamiltonia.

Let's use red numbers below in Table 9.1.4 to rank the fractional parts of the standard quotas from each state in order from largest to smallest. For example, Zeta's standard quota, 4.958, has the largest fractional part, 0.958. Also find the sum of the lower quotas to determine how many seats still need to be allocated.

Table 9.1.4: Fractional Parts for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532(6)	5.907(3)	2.954(2)	1.793(5)	6.857(4)	4.958(1)	25.001
Lower Quota	2	5	2	1	6	4	20

Twenty of the 25 seats have been allocated so there are five remaining seats. Allocate the seats, in order, to Zeta, Gamma, Beta, Epsilon and Delta.

Table 9.1.5: Final Allocation for Hamiltonia Using Hamilton's Method

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
Standard Quota	2.532	5.907	2.954	1.793	6.857	4.958	25.001
Lower Quota	2	5	2	1	6	4	20
Final Allocation	2	6	3	2	7	5	25

Overall, Alpha gets two senators, Beta gets six senators, Gamma gets three senators, Delta gets two senators, Epsilon gets seven, and Zeta gets five senators.

According to Ask.com, "a paradox is a statement that apparently contradicts itself and yet might be true." (Ask.com, 2014) Hamilton's method and the other apportionment methods discussed in section 9.2 are all subject to at least one paradox. None of the apportionment methods is perfect. The Alabama paradox was first noticed in 1881 when the seats in the U.S. House of Representatives were reapportioned after the 1880 census. At that time the U.S. Census Bureau created a table which showed the number of seats each state would have for various possible sizes of the House of Representatives. They did this for possible sizes of the House from 275 total seats to 350 total seats. This table showed a strange occurrence as the size of the House of Representatives increased from 299 to 300. With 299 total seats, Alabama would receive 8 seats. However, if the house size was increased to 300 total seats, Alabama would only receive 7 seats. Increasing the overall number of seats caused Alabama to lose a seat.

> The **Alabama paradox** happens when an increase in the total number of seats results in a decrease in the number of seats for a given state.

Example 9.1.5: The Alabama Paradox

A mother has an incentive program to get her five children to read more. She has 30 pieces of candy to divide among her children at the end of the week based on the number of minutes each of them spends reading. The minutes are listed below in Table 9.1.6.

Table 9.1.6: Reading Times

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750

Use Hamilton's method to apportion the candy among the children.

The standard divisor is $SD = \dfrac{750}{30} = 25$. After dividing each child's time by the standard divisor, and finding the lower quotas for each child, there are three pieces of candy left over. They will go to Ed, Bobby, and Dave, in that order, since they have the largest fractional parts of their quotas.

Table 9.1.7: Apportionment with 30 Pieces of Candy

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750
Standard Quota	7.520	5.680	5.520	2.560	8.720	30.000
Lower Quota	7	5	5	2	8	27
Final Allocation	7	6	5	3	9	30

At the last minute, the mother finds another piece of candy and does the apportionment again. This time the standard divisor will be 24.19. Bobby, Abby, and Charli, in that order, will get the three left over pieces this time.

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Table 9.1.8: Apportionment with 31 Pieces of Candy

Child	Abby	Bobby	Charli	Dave	Ed	Total
Population	188	142	138	64	218	750
Standard Quota	7.772	5.870	5.705	2.646	9.012	31.005
Lower Quota	7	5	5	2	9	28
Final Allocation	8	6	6	2	9	31

Notice that adding another piece of candy (a seat) caused Dave to lose a piece while Abby and Charli gain a piece. This is an example of the Alabama paradox.

Section 9.2: Apportionment: Jefferson's, Adams's, and Webster's Methods

Jefferson's method was the first method used to apportion the seats in the U.S. House of Representatives in 1792. It was used through 1832. That year, New York had a standard quota of 38.59 but was granted 40 seats by Jefferson's method. At that time, John Quincy Adams and Daniel Webster each proposed new apportionment methods but the proposals were defeated and Jefferson's method was still used. Webster's method was later chosen to be used in 1842 but Adams's method was never used. Webster's method and Hamilton's method often give the same result. For many of the years between 1852 and 1901, Congress used a number of seats for the House that would result in the same apportionment by either Webster's or Hamilton's methods. After Hamilton's method was finally scrapped in 1901, Webster's method was used in 1901, 1911, and 1931. There were irregularities in the process in 1872 and just after the 1920 census. In 1941, the House size was fixed at 435 seats and the Huntington-Hill method became the permanent method of apportionment.

Jefferson's, Adams's, and Webster's methods are all based on the idea of finding a divisor that will apportion all the seats under the appropriate rounding rule. There should be no seats left over after the number of seats are rounded off. For this to happen we have to adjust the standard divisor either up or down. The difference between the three methods is the rule for rounding off the quotas. Jefferson's method rounds the quotas down to their lower quotas, Adams' method rounds the quotas up to their upper quotas, and Webster's method rounds the quotas either up or down following the usual rounding rule.

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Jefferson's Method:

Jefferson's method divides all populations by a modified divisor and then rounds the results down to the lower quota. Sometimes the total number of seats will be too large and other times it will be too small. We keep guessing modified divisors until the method assigns the correct total number of seats. Dividing by a larger modified divisor will make each quota smaller so the sum of the lower quotas will be smaller. It is easy to remember which way to go. If the sum is too large, make the divisor larger. If the sum is too small, make the divisor smaller. All the quotas are rounded down so using the standard divisor will give a sum that is too small. Our guess for the first modified divisor should be a number smaller than the standard divisor.

Summary of Jefferson's Method:

1. Find the standard divisor, $SD = \dfrac{\text{total population}}{\text{\# seats}}$.
2. Pick a modified divisor, d, that is slightly less than the standard divisor.
3. Divide each state's population by the modified divisor to get its modified quota.
4. Round each modified quota down to its lower quota.
5. Find the sum of the lower quotas.
6. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too large, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps three through six until the correct number of seats are apportioned.

Example 9.2.1: Jefferson's Method

Use Jefferson's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480 and the sum of the lower quotas is 20. In Jefferson's method the standard divisor will always give us a sum that is too small so we begin by making the standard divisor smaller. There is no formula for this, just guess something. Let's try the modified divisor, d = 9000.

Table 9.2.2: Quotas for d = 9000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24

The sum of 24 is too small so we need to try again by making the modified divisor smaller. Let's try d = 8000.

Table 9.2.3: Quotas for d = 8000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24
d = 8000	3.00	7.00	3.50	2.13	8.13	5.88	
Lower Quota	3	7	3	2	8	5	28

This time the sum of 28 is too big. Try again making the modified divisor larger. We know the divisor must be between 8000 and 9000 so let's try 8500.

Table 9.2.4: Quotas for d = 8500

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9000	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24
d = 8000	3.00	7.00	3.50	2.13	8.13	5.88	
Lower Quota	3	7	3	2	8	5	28
d = 8500	2.82	6.59	3.29	2.00	7.65	5.53	
Lower Quota	2	6	3	2	7	5	25

This time the sum is 25 so we are done. Alpha gets two senators, Beta gets six senators, Gamma gets three senators, Delta gets two senators, Epsilon gets seven senators, and Zeta gets five senators.

Note: This is the same result as we got using Hamilton's method in Example 9.1.4. The two methods do not always give the same result.

Adams's Method:

Adams's method divides all populations by a modified divisor and then rounds the results up to the upper quota. Just like Jefferson's method we keep guessing modified divisors until the method assigns the correct number of seats. All the quotas are rounded up so the standard divisor will give a sum that is too large. Our guess for the first modified divisor should be a number larger than the standard divisor.

Summary of Adams's Method:

1. Find the standard divisor, $SD = \dfrac{\text{total population}}{\text{\# seats}}$.
2. Pick a modified divisor, d, that is slightly more than the standard divisor.
3. Divide each state's population by the modified divisor to get the modified quota.
4. Round each modified quota up to the upper quota.
5. Find the sum of the upper quotas.
6. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps three through six until the correct number of seats are apportioned.

Example 9.2.2: Adams's Method

Use Adams's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.5: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480 and the sum of the upper quotas is 26. In Adams's method the standard divisor will always give us a sum that is too big so we begin by making the standard divisor larger. There is no formula for this, just guess something. Let's try the modified divisor, d = 10,000.

Table 9.2.6: Quotas for d = 10,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Upper Quota	3	6	3	2	7	5	26

The total number of seats, 26, is too big so we need to try again by making the modified divisor larger. Try d = 11,000.

Table 9.2.7: Quotas for d = 11,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Upper Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Upper Quota	3	6	3	2	6	5	25

This time the total number of seats is 25, the correct number of seats to be apportioned. Give Alpha three seats, Beta six seats, Gamma three seats, Delta two seats, Epsilon six seats, and Zeta five seats.

Note: This is not the same result as we got using Hamilton's method in Example 9.1.4.

Webster's Method:

Webster's method divides all populations by a modified divisor and then rounds the results up or down following the usual rounding rules. Just like Jefferson's method we keep guessing modified divisors until the method assigns the correct number of seats. Because some quotas are rounded up and others down we do not know if the standard divisor will give a sum that is too large or too small. Our guess for the first modified divisor should be the standard divisor.

Summary of Webster's Method:

1. Find the standard divisor, $SD = \dfrac{\text{total population}}{\text{\# seats}}$. Use the standard divisor as the first modified divisor.
2. Divide each state's population by the modified divisor to get the modified quota.
3. Round each modified quota to the nearest integer using conventional rounding rules.
4. Find the sum of the rounded quotas.
5. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps two through five until the correct number of seats are apportioned.

Example 9.2.3: Webster's Method

Use Webster's method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.2.8: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

From Example 9.1.2 we know the standard divisor is 9480. Because some quotas will be rounded up and other quotas will be rounded down we do not know immediately whether the total number of seats is too big or too small. Unlike Jefferson's and Adam's method, we do not know which way to adjust the modified divisor. This forces us to use the standard divisor as the first modified divisor.

Note that we must use more decimal places in this example than in the last few examples. Using two decimal places gives more information about which way to round correctly. Think about Alpha's standard quota. Both 2.48 and 2.53 would round off to 2.5. However, 2.48 should be rounded down to 2 while 2.53 should be rounded up to 3 according to Webster's method. This situation has not happened in any of the previous examples.

Table 9.2.9: Quotas for d = 9480

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26

Since the total of 26 seats is too big we need to make the modified divisor larger. Try d = 11,000.

Table 9.2.10: Quotas for d = 11,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22

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The total number of seats is smaller like we hoped but 22 is way too small. That means that d = 11,000 is much too big. We need to pick a new modified divisor between 9480 and 11,000. Try a divisor closer to 9480 such as d = 10,000.

Table 9.2.11: Quotas for d = 10,000

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26
d = 11,000	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22
d = 10,000	2.40	5.60	2.80	1.70	6.50	4.70	
Rounded Quota	2	6	3	2	7	5	25

Note: This is the same apportionment we found using Hamilton's and Jefferson's methods, but not Adam's method.

9.3: Huntington-Hill Method

The Huntington-Hill method is the method currently used to apportion the seats for the U.S. House of Representatives. As with the other apportionment methods, the method of rounding off the quotas is what distinguishes this method from the others. The Huntington-Hill method starts out similarly to Webster's method since some quotas are rounded up and some quotas are rounded down. The difference is that the cut-off for rounding is not 0.5 anymore. Now the cut-off depends on the geometric mean between the lower and upper quotas.

The **geometric mean** G of two positive numbers A and B is $G = \sqrt{AB}$

Example 9.3.1: Geometric Mean

Find the geometric mean between 5 and 6.

$$G = \sqrt{5 \cdot 6} = \sqrt{30} \approx 5.477$$

Note that the geometric mean between A and B must be a number between A and B. In this example the geometric mean of 5.477 is between 5 and 6.

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Summary of the Huntington-Hill Method:

1. Find the standard divisor, $SD = \dfrac{\text{total population}}{\text{\# seats}}$. Use the standard divisor as the first modified divisor.
2. Divide each state's population by the modified divisor to get the modified quota.
3. Round each modified quota to the nearest integer using the geometric mean as the cut off. If the quota is less than the geometric mean between the upper and lower quotas, round the quota down to the lower quota. If the quota is more than the geometric mean between the upper and lower quotas, round the quota up to the upper quota.
4. Find the sum of the rounded quotas.
5. If the sum is the same as the number of seats to be apportioned, you are done. If the sum is too big, pick a new modified divisor that is larger than d. If the sum is too small, pick a new modified divisor that is smaller than d. Repeat steps two through five until the correct number of seats are apportioned.

Example 9.3.2: Huntington-Hill Method

Use the Huntington-Hill method to apportion the 25 seats in Hamiltonia from Example 9.1.2.

Table 9.3.1: Populations by State for Hamiltonia

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000

The first step is to use the standard divisor as the first modified divisor. We also include a row for the geometric mean between the upper and lower quotas for each state.

Table 9.3.2: Quotas for d = 9480

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Lower Quota	2	5	2	1	6	4	
Upper Quota	3	6	3	2	7	5	
Calculation for G	$\sqrt{2 \cdot 3}$	$\sqrt{5 \cdot 6}$	$\sqrt{2 \cdot 3}$	$\sqrt{1 \cdot 2}$	$\sqrt{6 \cdot 7}$	$\sqrt{4 \cdot 5}$	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26

The total number of seats, 26, is too big so we need to try again by making the modified divisor larger. Try d = 10,500.

Table 9.3.3: Quotas for d = 10,500

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26
d = 10,500	2.29	5.33	2.67	1.62	6.19	4.48	
Geometric Mean	2.449	5.477	2.449	1.414	6.481	4.472	
Rounded Quota	2	5	3	2	6	5	23

The total number of seats, 23 is too small. We need to try again with a modified divisor between 9480 and 10,500. Since 23 is further from 25 than 26 is, try a divisor closer to 9480. Try d = 9800.

Table 9.3.4: Quotas for d = 9800

State	Alpha	Beta	Gamma	Delta	Epsilon	Zeta	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
d = 9480	2.53	5.91	2.95	1.79	6.86	4.96	
Geometric Mean	2.449	5.477	2.450	1.414	6.481	4.472	
Rounded Quota	3	6	3	2	7	5	26
d = 10,500	2.29	5.33	2.67	1.62	6.19	4.48	
Geometric Mean	2.449	5.480	2.450	1.410	6.480	4.470	
Rounded Quota	2	5	3	2	6	5	23
d = 9800	2.4490	5.71	2.86	1.73	6.63	4.80	
Geometric Mean	2.4495	5.480	2.450	1.410	6.480	4.470	
Rounded Quota	2	6	3	2	7	5	25

Note: It was necessary to use more decimal places for Alpha's quota than the other quotas in order to see which way to round off.

This is the same apportionment we got with most of the other methods.

Example 9.3.3: Comparison of all Apportionment Methods

In the city of Adamstown, 42 new firefighters have just completed their training. They are to be assigned to the five firehouses in town in a manner proportional to

the population in each fire district. The populations are listed in the following table.

Table 9.3.5: Populations for the Fire Districts of Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465

Apportion the new firefighters to the fire houses using Hamilton's, Jefferson's, Adams's, Webster's, and Huntington-Hill's methods.

The standard divisor is $SD = \dfrac{69,465}{42} \approx 1654.$

Hamilton's Method:

Table 9.3.6: Hamilton's Method for Adamstown

Start by dividing each population by the standard divisor and rounding each standard quota down.

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
Standard Quota	15.121	5.296	7.007	5.456	9.117	41.998
Lower Quota	15	5	7	5	9	41
Final Allocation	15	5	7	6	9	42

Using the lower quotas, there is one firefighter left over. Assign this firefighter to District D since D has the largest fractional part.

Jefferson's Method:

Jefferson's method always rounds down making the sum of the lower quotas too small. Make the standard divisor smaller to get the first modified divisor. The results are summarized below in Table 9.3.7.

Guess #1: d = 1600. The sum of 41 is still too small so make the modified divisor smaller.

Guess #2: d = 1550. The sum is 42 so we are done.

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Table 9.3.7: Jefferson's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1600	15.631	5.475	7.244	5.641	9.425	
quota	15	5	7	5	9	41
d = 1550	16.135	5.652	7.477	5.823	9.729	
Final Allocation	16	5	7	5	9	42

Adams's Method:

Adams's method always rounds up making the sum of the upper quotas too large. Make the standard divisor larger to get the first modified divisor. The results are summarized below in Table 9.3.8.

Guess #1: d = 1700. The total is too still too large so make the modified divisor larger.

Guess #2: d = 1900. Now the total is too small so make the modified divisor smaller.

Guess #3: d = 1750. The total is too large again so make the modified divisor larger.

Guess #4: d = 1775. The sum is 42 so we are done.

Table 9.3.8: Adams's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1700	14.712	5.153	6.818	5.309	8.871	
quota	15	6	7	6	9	43
d = 1800	13.894	4.867	6.439	5.014	8.378	
quota	14	5	7	6	9	41
d = 1750	14.291	5.006	6.623	5.157	8.617	
quota	15	6	7	6	9	43
d = 1775	14.090	4.935	6.530	5.085	8.496	
Final Allocation	15	5	7	6	9	42

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Webster's Method:

Webster's method rounds the usual way so we cannot tell if the sum is too large or too small right away. Try the standard divisor as the first modified divisor. The results are summarized below in Table 9.3.9.

Guess #1: d = 1654. The sum of 41 is too small so make the modified divisor smaller.

Guess #2: d = 1600. The sum of 43 is too large so make the modified divisor larger.

Guess #3: d = 1625. The sum is 42 so we are done.

Table 9.3.9: Webster's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1654	15.121	5.296	7.007	5.456	9.117	
quota	15	5	7	5	9	41
d = 1600	15.631	5.475	7.244	5.641	9.425	
quota	16	5	7	6	9	43
d = 1625	15.391	5.391	7.132	5.554	9.280	
Final Allocation	15	5	7	6	9	42

Huntington-Hill's Method:

Huntington-Hill's method rounds off according to the geometric mean. Use the standard divisor as the first modified divisor. The results are summarized below in Table 9.3.10.

Guess #1: d = 1654. The sum of 41 is too small so make the modified divisor smaller. Look at District D. It was really close to being rounded up rather than rounded down so we do not need to change the modified divisor by very much.

Guess #2: d = 1625. The sum is 42 so we are done.

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Table 9.3.10: Huntington-Hill's Method for Adamstown

District	A	B	C	D	E	Total
Population	25,010	8,760	11,590	9,025	15,080	69,465
d = 1654	15.121	5.296	7.007	5.456	9.117	
Geometric mean	15.492	5.477	7.483	5.477	9.487	
quota	15	5	7	5	9	41
d = 1625	15.391	5.391	7.132	5.554	9.280	
Geometric mean	15.492	5.477	7.483	5.477	9.487	
Final Allocation	15	5	7	6	9	42

Hamilton's, Adams's, Webster's, and Huntington-Hill's methods all gave the same apportionment: 15 firefighters to District A, five to District B, seven to District C, six to District D, and nine to District E.

Jefferson's method gave a different apportionment: 16 firefighters to District A, five to District B, seven to District C, five to District D, and nine to District E.

9.4: Apportionment Paradoxes

Each of the apportionment methods has at least one weakness. Some potentially violate the quota rule and some are subject to one of the three paradoxes.

> The **quota rule** says that each state should be given either its upper quota of seats or its lower quota of seats.

Example 9.4.1: Quota Rule Violation

A small college has three departments. Department A has 98 faculty, Department B has 689 faculty, and Department C has 212 faculty. The college has a faculty senate with 100 representatives. Use Jefferson's method with a modified divisor of d = 9.83 to apportion the 100 representatives among the departments.

Table 9.4.1: Quota Rule Violation

State	A	B	C	Total
Population	98	689	212	999
Standard Quota	9.810	68.969	21.221	100.000
d = 9.83	9.969	70.092	21.567	
quota	9.000	70.000	21.000	100

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District B has a standard quota of 68.969 so it should get either its lower quota, 68, or its upper quota, 69, seats. Using this method, District B received 70 seats, one more than its upper quota. This is a Quota Rule violation.

> The **population paradox** occurs when a state's population increases but its allocated number of seats decreases.

Example 9.4.2: Population Paradox

A mom decides to split 11 candy bars among three children based on the number of minutes they spend on chores this week. Abby spends 54 minutes, Bobby spends 243 minutes and Charley spends 703 minutes. Near the end of the week, Mom reminds the children of the deal and they each do some extra work. Abby does an extra two minutes, Bobby an extra 12 minutes and Charley an extra 86 minutes. Use Hamilton's method to apportion the candy bars both before and after the extra work.

Table 9.4.2: Candy Bars Before the Extra Work

State	Abby	Bobby	Charley	Total
Population	54	243	703	1,000
Standard Quota	0.594	2.673	7.734	11.000
Lower Quota	0	2	7	9
Apportionment	0	3	8	11

With the extra work:
Abby now has 54 + 2 = 56 minutes
Bobby has 243 + 12 = 255
Charley has 703 + 86 = 789 minutes

Table 9.4.3: Candy Bars After the Extra Work

State	Abby	Bobby	Charley	Total
Population	56	255	789	1,100
Standard Quota	0.560	2.550	7.890	11.000
Lower Quota	0	2	7	9
Apportionment	1	2	8	11

Abby's time only increased by 3.7% while Bobby's time increased by 4.9%. However, Abby gained a candy bar while Bobby lost one. This is an example of the Population Paradox.

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> The **new states paradox** occurs when a new state is added along with additional seats and existing states lose seats.

Example 9.4.3: New-States Paradox

A small city is made up of three districts and governed by a committee with 100 members. District A has a population of 5310, District B has a population of 1330, and District C has a population of 3308. The city annexes a small area, District D with a population of 500. At the same time the number of committee members is increased by five. Use Hamilton's method to find the apportionment before and after the annexation.

$$SD = \frac{9948}{100} = 99.48$$

Table 9.4.4: Apportionment Before the Annexation

State	A	B	C	Total
Population	5,310	1,330	3,308	9,948
Standard Quota	53.378	13.370	33.253	100.000
Lower Quota	53	13	33	99
Apportionment	54	13	33	100

$$SD = \frac{10,448}{105} = 99.505$$

Table 9.4.5: Apportionment After the Annexation

State	A	B	C	D	Total
Population	5,310	1,330	3,308	500	10,448
Standard Quota	53.364	13.366	33.245	5.025	105.000
Lower Quota	53	13	33	5	104
Apportionment	53	14	33	5	105

District D has a population of 500 so it should get five seats. When District D is added with its five seats, District A loses a seat and District B gains a seat. This is an example of the New-States Paradox.

In 1980, Michael Balinski (State University of New York at Stony Brook) and H. Peyton Young (Johns Hopkins University) proved that all apportionment methods either violate the quota rule or suffer from one of the paradoxes. This means that it is impossible to find

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the "perfect" apportionment method. The methods and their potential flaws are listed in the following table.

Table 9.4.6: Methods, Quota Rule Violations, and Paradoxes

Method	Quota Rule	Paradoxes		
		Alabama	**Population**	**New-States**
Hamilton	No violations	Yes	Yes	Yes
Jefferson	Upper-quota violations	No	No	No
Adams	Lower-quota violations	No	No	No
Webster	Lower- and upper-quota violations	No	No	No
Huntington-Hill	Lower- and upper-quota violations	No	No	No

Chapter 9 Homework

1. A solar system consisting of five planets has a governing council of 135 members who are apportioned proportional to the populations of the planets. The population for each planet is listed in the following table.

Planet	Ajax	Borax	Calax	Delphi	Eljix	Total
Population	183,000	576,000	274,000	749,000	243,000	2,025,000

For each planet, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

2. A city has seven fire districts and 585 firefighters. The number of firefighters assigned to each district is proportional to the population of the district. The population for each district is given in the following table.

District	1	2	3	4	5	6	7	Total
Population	23,400	41,800	36,200	28,800	34,900	48,500	16,300	229,900

For each district, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

3. The country named Erau has five states and a total of 200 seats available in its House of Representatives. The number of seats that each state receives is proportional to the population of that state. The populations of the states are given in the table below.

State	1	2	3	4	5	Total
Population	3,500,000	1,200,000	530,000	999,000	771,000	7,000,000

For each state, find the standard quota, the upper quota and the lower quota. Give your answers in a table.

4. Use Hamilton's method to apportion the 585 firefighters in problem #2.

5. Use Hamilton's method to apportion the 135 council members in problem #1.

6. Use Hamilton's method to apportion the 200 seats in the House of Representatives in problem #3.

7. A small country is made up of three separate islands: Eno, with a population of 100,300, Owt, with a population of 9,405, and Eerht with a population of 90,295. The country has a senate with 200 members whose seats are apportioned proportional to the population of each island.
   a. Use Hamilton's method to apportion the 200 seats.
   b. The senate decides to add another seat so that they have an odd number of senators. Use Hamilton's method to apportion the 201 seats.
   c. Compare your results from parts (a) and (b). This is an example of which paradox?

8. Use Jefferson's method to apportion the 585 firefighters in problem #2.

9. Use Jefferson's method to apportion the 135 council members in problem #1.

10. Use Jefferson's method to apportion the 200 seats in the House of Representatives in problem #3.

11. Use Adams's method to apportion the 135 council members in problem #1.

12. Use Adams's method to apportion the 585 firefighters in problem #2.

13. Use Adam's method to apportion the 200 seats in the House of Representatives in problem #3.

14. Use Webster's method to apportion the 585 firefighters in problem #2.

15. Use Webster's method to apportion the 135 council members in problem #1.

16. Use Webster's method to apportion the 200 seats in the House of Representatives in problem #3.

17. Use Huntington-Hill's method to apportion the 135 council members in problem #1.

18. Use Huntington-Hill's method to apportion the 585 firefighters in problem #2.

19. Use Huntington-Hill's method to apportion the 200 seats in the House of Representatives in problem #3.

20. Last year a city had three school districts: North, with a population of 5,200 children, South, with a population of 10,600 children, and West, with a population of 15,100.
    a. Use Hamilton's method to apportion 50 speech therapists among the districts using the populations for last year.
    b. This year, the city took over another school district. The new East district has a population of 9,500 children. If the number of speech therapists is increased by 15 to accommodate the new district, use Hamilton's method to apportion the 65 speech therapists.
    c. Compare your results to parts (a) and (b). This is an example of which paradox?

21. After the census in 1950, planet Ajax had a standard quota of 11.87 but was awarded 13 seats using Jefferson's method. Why were people on the other planets upset about this? What rule was violated?

22. Five co-workers work together on a large design project. As a reward for their effort, their boss wishes to split 50 gift cards of equal value among the employees based on the amount of time spend on the project. The time, in hours, worked by each employee is listed in the following table.

Employee	Jack	Kim	Lisa	Mark	Nancy	Total
Time	150	173	78	295	204	900

   a. Use Hamilton's method to apportion the 50 gift cards among the employees.
   b. At the last minute, the boss made a minor change to the design. It took Kim 8 more hours and Mark one more hour to incorporate the change. Use Hamilton's method to apportion the 50 gift cards using the new total times for Kim (181) and Mark (296).
   c. Compare your answers to parts (a) and (b). Which one of the paradoxes occurred? Explain your answer.