Chapter 3: Probability

We see probabilities almost every day in our real lives. Most times you pick up the newspaper or read the news on the internet, you encounter probability. There is a 65% chance of rain today, or a pre-election poll shows that 52% of voters approve of a ballot

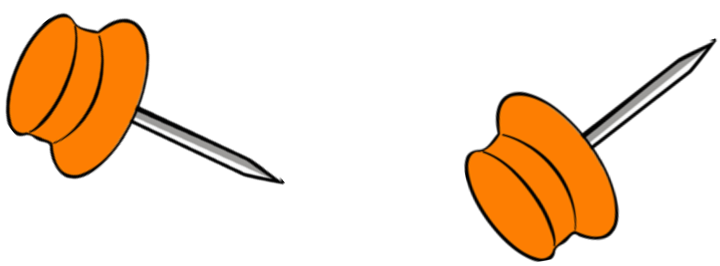
of the ideas students think they know about probability are incorrect. This is one area of item, are examples of probabilities. Did you ever wonder why a flush beats a full house in poker? It's because the probability of getting a flush is smaller than the probability of getting a full house. Probabilities can also be used to make business decisions, figure out insurance premiums, and determine the price of raffle tickets.

If an experiment has only three possible outcomes, does this mean that each outcome has a 1/3 chance of occurring? Many students who have not studied probability would answer yes. Unfortunately, they could be wrong. The answer depends on the experiment. Many math where their intuition is sometimes misleading. Students need to use experiments or mathematical formulas to calculate probabilities correctly.

Section 3.1: Basic Probabilities and Probability Distributions; Three Ways to Define Probabilities

Toss a thumb tack one time. Do you think the tack will land with the point up or the point down?

Figure 3.1.1: Which Way Will a Tack Fall?



(Thumbtack, n.d.)

We cannot predict which way the tack will land before we toss it. Sometimes it will land with the point up and other times it will land with the point down. Tossing a tack is a random experiment since we cannot predict what the outcome will be. We do know that there are only two possible outcomes for each trial of the experiment: lands points up or lands point down. If we repeat the experiment of tossing the tack many times we might be able to guess how likely it is that the tack will land point up.

A **random experiment** is an activity or an observation whose outcome cannot be predicted ahead of time

A **trial** is one repetition of a random experiment.

The **sample space** is the set of all possible outcomes for a random experiment.

An **event** is a subset of the sample space.

Do you think chances of the tack landing point up and the tack landing point down are the same? This is an example where your intuition may be wrong. Having only two possible outcomes does not mean each outcome has a 50/50 chance of happening. In fact we are going to see that the probability of the tack landing point up is about 66%.

To begin to answer this question, toss a tack 10 times. For each toss record whether the tack lands point up or point down.

Table 3.1.2: Toss a Tack Ten Times

Trial	1	2	3	4	5	6	7	8	9	10
Up/Down	Up	Down	Up	Up	Down	Up	Up	Down	Up	Down

The random experiment here is tossing the tack one time. The possible outcomes for the experiment are that the tack lands point up or the tack lands point down so the sample space is S = {point up, point down}. We are interested in the event E that the tack lands point up, E = {point up}.

Based on our data we would say that the tack landed point up six out of 10 times. The fraction, $\frac{6}{10}$, is called the relative frequency. Since $\frac{6}{10} = 0.60$ we would guess that probability of the tack landing point up is about 60%.

Let’s repeat the experiment by tossing the tack ten more times.

Table 3.1.3: Toss a Tack Ten More Times

Trial	1	2	3	4	5	6	7	8	9	10
Up/Down	Up	Up	Up	Down	Down	Up	Up	Down	Up	Up

This time the tack landed point up seven out of 10 times or 70% of the time. If we tossed the tack another 10 times we might get a different result again. The probability of the tack landing point up refers to what happens when we toss the tack many, many times. Let’s toss the tack 150 times and count the number of times it lands point up. Along the way we will look at the proportion of landing point up.

Table 3.1.4: Toss a Tack Many Times

Trials	Number Up	Total Number Of Up	Total Number Of Trials	Proportion
0-10	6	6	10	6/10=0.60
11-20	7	13	20	13/20=0.65
21-30	8	21	30	21/30=0.70
31-40	6	27	40	27/40=0.68
41-50	8	35	50	35/50=0.70
51-60	8	43	60	43/60=0.72
61-70	4	47	70	47/70=0.67
71-80	7	54	80	54/80=0.68
81-90	5	59	90	59/90=0.66
91-100	6	65	100	65/100=0.65
101-110	10	75	110	75/110=0.68
111-120	5	80	120	80/120=0.67
121-130	8	88	130	88/130=0.68
131-140	4	92	140	92/140=0.66
141-150	7	99	150	99/150=0.66

When we have a small number of trials the proportion varies quite a bit. As we start to have more trials the proportion still varies but not by as much. It appears that the proportion is around 0.66 or 66%. We would have to do about 100,000 trials to get a better approximation of the actual probability of the tack landing point up.

The tack landed point up 99 out of 150 trials. The probability P of event E is written as:

$$P(E)=\frac{\#\text{ of trials with point up}}{\text{total number of trials}}=\frac{99}{150}\approx 0.66$$

We would say the probability that the tack lands point up is about 66%.

Equally Likely Outcomes:

In some experiments all the outcomes have the same chance of happening. If we roll a fair die the chances are the same for rolling a two or rolling a five. If we draw a single card from a well shuffled deck of cards, each card has the same chance of being selected. We call outcomes like these equally likely. Drawing names from a hat or drawing straws are other examples of equally likely outcomes. The tack tossing example did not have equally likely outcomes since the probability of the tack landing point up is different than the probability of the tack landing point down.

An experiment has **equally likely outcomes** if every outcome has the same probability of occurring.

For equally likely outcomes, the **probability of outcome A**, *P(A)*, is:

$$P(A) = \frac{\text{number of ways for A to occur}}{\text{total number of outcomes}}.$$

Round Off Rule: Give probabilities as a fraction or as a decimal number rounded to three decimal places.

Figure 3.1.5: Deck of Cards

(Pine, 2007)

Example 3.1.1: Simple Probabilities with Cards

Draw a single card from a well shuffled deck of 52 cards. Each card has the same chance of being drawn so we have equally likely outcomes. Find the following probabilities:

a. *P*(card is red)

$$P(\text{card is red}) = \frac{\text{number of red cards}}{\text{total number of cards}} = \frac{26}{52} = \frac{1}{2}$$

The probability that the card is red is $\frac{1}{2}$.

b. P(card is a heart)

$$P(\text{card is a heart}) = \frac{\text{number of hearts}}{\text{total number of cards}} = \frac{13}{52} = \frac{1}{4}$$

The probability that the card is a heart is $\frac{1}{4}$.

c. P(card is a red 5)

$$P(\text{card is a red 5}) = \frac{\text{number of red fives}}{\text{total number of cards}} = \frac{2}{52} = \frac{1}{26}$$

The probability that the card is a red five is $\frac{1}{26}$.

Example 3.1.2: Simple Probabilities with a Fair Die

Roll a fair die one time. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Find the following probabilities.

a. P(roll a four)

$$P(\text{roll a four}) = \frac{\text{number of ways to roll a four}}{\text{total number of ways to roll a die}} = \frac{1}{6}$$

The probability of rolling a four is $\frac{1}{6}$.

b. P(roll an odd number)

The event roll an odd number is E = {1, 3, 5}.

$$P(\text{roll an odd number}) = \frac{\text{number of ways to roll an odd number}}{\text{total number of ways to roll a die}} = \frac{3}{6} = \frac{1}{2}$$

The probability of rolling an odd number is $\frac{1}{2}$.

c. P(roll a number less than five)

The event roll a number less than five is F = {1, 2, 3, 4}.

$$P(\text{roll a number less than five}) = \frac{\text{number of ways to roll number less than five}}{\text{total number of ways to roll a die}} = \frac{4}{6} = \frac{2}{3}$$

The probability of rolling a number less than five is $\frac{2}{3}$.

Example 3.1.3: Simple Probability with Books

Figure 3.1.6: Books on a Shelf

(Bookshelf, 2011)

A small bookcase contains five math, three English and seven science books. A book is chosen at random. What is the probability that a math book is chosen.

Since the book is chosen at random each book has the same chance of being chosen and we have equally likely events.

$$P(\text{math book}) = \frac{\text{number of ways to choose a math book}}{\text{total number of books}} = \frac{5}{15} = \frac{1}{3}$$

The probability a math book was chosen is $\frac{1}{3}$.

Three Ways of Finding Probabilities:

There are three ways to find probabilities. In the tack tossing example we calculated the probability of the tack landing point up by doing an experiment and recording the outcomes. This was an example of an empirical probability. The probability of getting a red jack in a card game or rolling a five with a fair die can be calculated from mathematical formulas. These are examples of theoretical probabilities. The third type of probability is a subjective probability. Saying that there is an 80% chance that you will go to the beach this weekend is a subjective probability. It is based on experience or guessing.

A **theoretical probability** is based on a mathematical model where all outcomes are equally likely to occur.

An **empirical probability** is based on an experiment or observation and is the relative frequency of the event occurring.

A **subjective probability** is an estimate (a guess) based on experience or intuition.

Complements:

If there is a 75% chance of rain today, what are the chances it will not rain? We know that there are only two possibilities. It will either rain or it will not rain. Because the sum of the probabilities for all the outcomes in the sample space must be 100% or 1.00, we know that

P(will rain) + P(will not rain) = 100%.

Rearranging this we see that

P(will not rain) = 100% - P(will rain) = 100% - 75% = 25%.

The events E = {will rain} and F = {will not rain} are called complements.

The **complement** of event E, denoted by $\overline{E}$, is the set of outcomes in the sample space that are not in the event E. The probability of $\overline{E}$ is given by $P\left(\overline{E}\right) = 1 - P\left(E\right)$.

Example 3.1.4: Complements with Cards

Draw a single card from a well shuffled deck of 52 cards.

a. Look at the suit of the card. Here the sample space S = {spades, clubs, hearts, diamonds}. If event E = {spades} the complement $\overline{E}$ = {clubs, hearts, diamonds}.

b. Look at the value of the cards. Here the sample space is S = {A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K}. If the event E = {the number is less than 7} = {A, 2, 3, 4, 5, 6} the complement $\overline{E}$ = {7, 8, 9, 10, J, Q, K}.

Example 3.1.5: Complements with Trains

A train arrives on time at a particular station 85% of the time. Does this mean that the train is late 15% of the time? The answer is no. The complement of E = {on time} is not $\overline{E}$ = {late}. There is a third possibility. The train could be early. The sample space is S = {on time, early, late} so the complement of E = {on time} is $\overline{E}$ = {early or late}. Based on the given information we cannot find P(late) but we can find P(early or late) = 15%.

Impossible Events and Certain Events:

Recall that $P(A)=\frac{\text{number of ways for A to occur}}{\text{total number of outcomes}}$. What does it mean if we say the probability of the event is zero? $P(A)=\frac{\text{number of ways for A to occur}}{\text{total number of outcomes}}=0$. The only way for a fraction to equal zero is when the numerator is zero. This means there is no way for event A to occur. A probability of zero means that the event is impossible.

What does it mean if we say the probability of an event is one?
$P(A)=\frac{\text{number of ways for A to occur}}{\text{total number of outcomes}}=1$. The only way for a fraction to equal one is if the numerator and denominator are the same. The number of ways for A to occur is the same as the number of outcomes. There are no outcomes where A does not occur. A probability of 1 means that the event always happens.

$$P(A)=0 \text{ means that } A \text{ is impossible}$$
$$P(A)=1 \text{ means that } A \text{ is certain}$$

Probability Distributions:

A probability distribution (probability space) is a sample space paired with the probabilities for each outcome in the sample space. If we toss a fair coin and see which side lands up, there are two outcomes, heads and tails. Since the coin is fair these are equally likely outcomes and have the same probabilities. The probability distribution would be P(heads) = 1/2 and P(tails) = 1/2. This is often written in table form:

Table 3.1.7: Probability Distribution for a Fair Coin

Outcome	Heads	Tails
Probability	1/2	1/2

A **probability distribution** for an experiment is a list of all the possible outcomes and their corresponding probabilities.

Example 3.1.6: Probabilities for the Sum of Two Fair Dice

In probability problems when we roll two dice, it is helpful to think of the dice as being different colors. Let's assume that one die is red and the other die is green. We consider getting a three on the red die and a five on the green die different than getting a five on the red die and a three on the green die. In other words, when we list the outcomes the order matters. The possible outcomes of rolling two dice and looking at the sum are given in Table 3.1.8.

Table 3.1.8: All Possible Sums of Two Dice

1+1 = 2	1+2 = 3	1+3 = 4	1+4 = 5	1+5 = 6	1+6 = 7
2+1 = 3	2+2 = 4	2+3 = 5	2+4 = 6	2+5 = 7	2+6 = 8
3+1 = 4	3+2 = 5	3+3 = 6	3+4 = 7	3+5 = 8	3+6 = 9
4+1 = 5	4+2 = 6	4+3 = 7	4+4 = 8	4+5 = 9	4+6 = 10
5+1 = 6	5+2 = 7	5+3 = 8	5+4 = 9	5+5 = 10	5+6 = 11
6+1 = 7	6+2 = 8	6+3 = 9	6+4 = 10	6+5 = 11	6+6 = 12

Table 3.1.9: Probability Distribution for the Sum of Two Fair Dice

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
Reduced Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Example 3.1.7: Valid and Invalid Probability Distributions

Are the following valid probability distributions?

a. **Table 3.1.10:**

Outcome	A	B	C	D	E
Probability	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

This is a valid probability distribution. All the probabilities are between zero and one inclusive and the sum of the probabilities is 1.00.

b. **Table 3.1.11:**

Outcome	A	B	C	D	E	F
Probability	0.45	0.80	-0.20	-0.35	0.10	0.20

This is not a valid probability distribution. The sum of the probabilities is 1.00, but some of the probabilities are not between zero and one, inclusive.

c. **Table 3.1.12:**

Outcome	A	B	C	D
Probability	0.30	0.20	0.40	0.25

This is not a valid probability distribution. All of the probabilities are between zero and one, inclusive, but the sum of the probabilities is 1.15 not 1.00.

Odds:

Probabilities are always numbers between zero and one. Many people are not comfortable working with such small values. Another way of describing the likelihood of an event happening is to use the ratio of how often it happens to how often it does not happen. The ratio is called the odds of the event happening. There are two types of odds, odds for and odds against. Casinos, race tracks and other types of gambling usually state the odds against an event happening.

If the probability of an event *E* is *P(E)*, then the **odds for event *E***, *O(E)*, are given by:

$$O(E)=\frac{P(E)}{P(\bar{E})} \qquad \text{OR} \qquad O(E)=\frac{\text{number of ways for E to occur}}{\text{number of ways for E to not occur}}$$

Also, the **odds against event *E***, are given by:

$$O(\bar{E})=\frac{P(\bar{E})}{P(E)} \qquad \text{OR} \qquad O(E)=\frac{\text{number of ways for E to not occur}}{\text{number of ways for E to occur}}$$

Example 3.1.8: Odds in Drawing a Card

A single card is drawn from a well shuffled deck of 52 cards. Find the odds that the card is a red eight.

There are two red eights in the deck.

$$P(\text{red eight})=\frac{2}{52}=\frac{1}{26}$$

$$P(\text{not a red eight})=\frac{50}{52}=\frac{25}{26}.$$

$$O(\text{red eight}) = \frac{P(\text{red eight})}{P(\text{not a red eight})} = \frac{\frac{1}{26}}{\frac{25}{26}} = \frac{1}{\cancel{26}} \cdot \frac{\cancel{26}}{25} = \frac{1}{25}.$$

The odds of drawing a red eight are 1 to 25. This can also be written as 1:25.

Note: Do not write odds as a decimal or a percent.

Example 3.1.9: Odds in Roulette

Many roulette wheels have slots numbered 0, 00, and 1 through 36. The slots numbered 0 and 00 are green. The even numbered slots are red and the odd numbered slots are black. The game is played by spinning the wheel one direction and rolling a marble around the outer edge the other direction. Players bet on which slot the marble will fall into. What are the odds the marble will land in a red slot?

There are 38 slots in all. The slots 2, 4, 6, …, 36 are red so there are 18 red slots. The other 20 slots are not red.

$$P(\text{red}) = \frac{18}{38} = \frac{9}{19}$$

$$P(\text{not red}) = 1 - \frac{9}{19} = \frac{19}{19} - \frac{9}{19} = \frac{10}{19}$$

$$O(\text{red}) = \frac{P(\text{red})}{P(\text{not red})} = \frac{\frac{9}{19}}{\frac{10}{19}} = \frac{9}{\cancel{19}} \cdot \frac{\cancel{19}}{10} = \frac{9}{10}$$

The odds of the marble landing in a red slot are 9 to 10. This can also be written as 9:10.

Example 3.1.10: Odds Against an Event

Two fair dice are tossed and the sum is recorded. Find the odds against rolling a sum of nine.

The event *E*, roll a sum of nine is: *E* = {(3, 6), (4, 5), (5, 4), (6, 3)}

There are 36 ways to roll two dice and four ways to roll a sum of nine. That means there are 32 ways to roll a sum that is not nine.

$$P(\text{sum is nine}) = \frac{4}{36} = \frac{1}{9}$$

$$P(\text{sum is not nine}) = \frac{32}{36} = \frac{8}{9}$$

$$O(\text{against sum is nine}) = \frac{P(\text{sum is not nine})}{P(\text{sum is nine})} = \frac{\frac{8}{9}}{\frac{1}{9}} = \frac{8}{\cancel{9}} \cdot \frac{\cancel{9}}{1} = \frac{8}{1}$$

The odds against rolling a sum of nine are 8 to 1 or 8:1.

We can also find the probability of an event happening based on the odds for the event. Saying that the odds of an event are 3 to 5 means that the event happens three times for every five times it does not happen. If we add up the possibilities of both we get a sum of eight. So the event happens about three out of every eight times. We would say the probability is 3/8.

If the odds favoring event E are a to b, then: $P(E) = \frac{a}{a+b}$ and $P(\bar{E}) = \frac{b}{a+b}$.

Example 3.1.11: Finding the Probability from the Odds

A local little league baseball team is going to a tournament. The odds of the team winning the tournament are 3 to 7. Find the probability of the team winning the tournament.

$$P(\text{winning}) = \frac{3}{3+7} = \frac{3}{10} = 0.3$$

Section 3.2: Combining Probabilities with "And" and "Or"

Many probabilities in real life involve more than one outcome. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two outcomes to make a single event we connect the outcomes with the word "and" or the word "or." It is very important in probability to pay attention to the words "and" and "or" if they appear in a problem. The word "and" restricts the field of possible outcomes to only those outcomes that simultaneously satisfy more than one event. The word "or" broadens the field of possible outcomes to those that satisfy one or more events.

Example 3.2.1: Counting Students

Figure 3.2.1: College Classroom

(Colwell, 2013)

Suppose a teacher wants to know the probability that a single student in her class of 30 students is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates

$$P(\text{Art or English}) = \frac{13+21}{30} = \frac{33}{30}.$$

The teacher knows that this is wrong because probabilities must be between zero and one, inclusive. After thinking about it she remembers that nine students are taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities we have to be careful to count each outcome only once.

Mutually Exclusive Events:

An experiment consists of drawing one card from a well shuffled deck of 52 cards. Consider the events E: the card is red, F: the card is a five, and G: the card is a spade. It is possible for a card to be both red and a five at the same time but it is not possible for a card to be both red and a spade at the same time. It would be easy to accidentally count a red five twice by mistake. It is not possible to count a red spade twice.

Two events are **mutually exclusive** if they have no outcomes in common.

Example 3.2.2: Mutually Exclusive with Dice

Two fair dice are tossed and different events are recorded. Let the events *E*, *F* and *G* be as follows:

E = {the sum is five} = {(1, 4), (2, 3), (3, 2), (4, 1)}

F = {both numbers are even} = {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}

G = {both numbers are less than five} = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4,1), (4, 2), (4, 3), (4,4)}

a. Are events *E* and *F* mutually exclusive?

   Yes. *E* and *F* are mutually exclusive because they have no outcomes in common. It is not possible to add two even numbers to get a sum of five.

b. Are events *E* and *G* mutually exclusive?

   No. *E* and *G* are not mutually exclusive because they have some outcomes in common. The pairs (1, 4), (2, 3), (3, 2) and (4, 1) all have sums of 5 and both numbers are less than five.

c. Are events *F* and *G* mutually exclusive?

   No. *F* and *G* are not mutually exclusive because they have some outcomes in common. The pairs (2, 2), (2, 4), (4, 2) and (4, 4) all have two even numbers that are less than five.

Addition Rule for "Or" Probabilities:

The addition rule for probabilities is used when the events are connected by the word "or". Remember our teacher in Example 3.2.1 at the beginning of the section? She wanted to know the probability that her students were taking either art or English. Her problem was that she counted some students twice. She needed to add the number of students taking art to the number of students taking English and then subtract the number of students she counted twice. After dividing the result by the total number of students she will find the desired probability. The calculation is as follows:

$$P(\text{art or English}) = \frac{\text{\# taking art + \# taking English - \# taking both}}{\text{total number of students}}$$
$$= \frac{13+21-9}{30}$$
$$= \frac{25}{30} \approx 0.833$$

The probability that a student is taking art or English is 0.833 or 83.3%.

When we calculate the probability for compound events connected by the word "or" we need to be careful not to count the same thing twice. If we want the probability of drawing a red card or a five we cannot count the red fives twice. If we want the probability a person is blonde-haired or blue-eyed we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people then subtracts the number of people we counted twice.

Addition Rule for "Or" Probabilities
If *A* and *B* are any events then, *P(A* or *B) = P(A) + P(B) – P(A* and *B).*
If *A* and *B* are mutually exclusive events then *P(A* and *B) = 0*, so then
P(A or *B) = P(A) + P(B)*.

Example 3.2.3: Additional Rule for Drawing Cards

A single card is drawn from a well shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

There are 13 cards that are clubs, 12 face cards (J, Q, K in each suit) and 3 face cards that are clubs.

$$P(\text{club or face card}) = P(\text{club}) + P(\text{face card}) - P(\text{club and face card})$$
$$= \frac{13}{52} + \frac{12}{52} - \frac{3}{52}$$
$$= \frac{22}{52} = \frac{11}{26} \approx 0.423$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

Example 3.2.4: Addition Rule for Tossing a Coin and Rolling a Die

An experiment consists of tossing a coin then rolling a die. Find the probability that the coin lands heads up or the number is five.

Let H represent heads up and T represent tails up. The sample space for this experiment is S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

There are six ways the coin can land heads up, {H1, H2, H3, H4, H5, H6}.

There are two ways the die can land on five, {H5, T5}.

There is one way for the coin to land heads up and the die to land on five, {H5}.

$$\begin{aligned} P(\text{heads or five}) &= P(\text{heads}) + P(\text{five}) - P(\text{both heads and five}) \\ &= \frac{6}{12} + \frac{2}{12} - \frac{1}{12} \\ &= \frac{7}{12} \approx 0.583 \end{aligned}$$

The probability that the coin lands heads up or the number is five is approximately 0.583 or 58.3%.

Example 3.2.5: Addition Rule for Satisfaction of Car Buyers

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Find the probability that a person bought a new car or was not satisfied.

$$\begin{aligned} P(\text{new car or not satisfied}) &= P(\text{new car}) + P(\text{not satisfied}) - P(\text{new car and not satisfied}) \\ &= \frac{120}{250} + \frac{75}{250} - \frac{28}{250} = \frac{167}{250} \approx 0.668 \end{aligned}$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

Independent Events:

Sometimes we need to calculate probabilities for compound events that are connected by the word "and." We have two methods to choose from, independent events or conditional probabilities (Section 3.3). Tossing a coin multiple times or rolling dice are independent events. Each time you toss a fair coin the probability of getting heads is ½. It does not matter what happened the last time you tossed the coin. It's similar for dice. If you rolled double sixes last time that does not change the probability that you will roll double sixes this time. Drawing two cards without replacement is not an independent event. When you draw the first card and set it aside, the probability for the second card is now out of 51 cards not 52 cards.

Two events are **independent events** if the occurrence of one event has no effect on the probability of the occurrence of the other event.

Multiplication Rule for "And" Probabilities: Independent Events

If events A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Example 3.2.6: Independent Events for Tossing Coins

Suppose a fair coin is tossed four times. What is the probability that all four tosses land heads up?

The tosses of the coins are independent events. Knowing a head was tossed on the first trial does not change the probability of tossing a head on the second trial.

$$\begin{aligned} P(\text{four heads in a row}) &= P(\text{1st heads and 2nd heads and 3rd heads and 4th heads}) \\ &= P(\text{1st heads}) \cdot P(\text{2nd heads}) \cdot P(\text{3rd heads}) \cdot P(\text{4th heads}) \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{16} \end{aligned}$$

The probability that all four tosses land heads up is $\frac{1}{16}$.

Example 3.2.7: Independent Events for Drawing Marbles

A bag contains five red and four white marbles. A marble is drawn from the bag, its color recorded and the marble is returned to the bag. A second marble is then drawn. What is the probability that the first marble is red and the second marble is white?

Since the first marble is put back in the bag before the second marble is drawn these are independent events.

$$
\begin{aligned}
P(\text{1st red and 2nd white}) &= P(\text{1st red}) \cdot P(\text{2nd white}) \\
&= \frac{5}{9} \cdot \frac{4}{9} = \frac{20}{81}
\end{aligned}
$$

The probability that the first marble is red and the second marble is white is $\frac{20}{81}$.

Example 3.2.8: Independent Events for Faulty Alarm Clocks

Abby has an important meeting in the morning. She sets three battery-powered alarm clocks just to be safe. If each alarm clock has a 0.03 probability of malfunctioning, what is the probability that all three alarm clocks fail at the same time?

Since the clocks are battery powered we can assume that one failing will have no effect on the operation of the other two clocks. The functioning of the clocks is independent.

$$
\begin{aligned}
P(\text{all three fail}) &= P(\text{first fails}) \cdot P(\text{second fails}) \cdot P(\text{third fails}) \\
&= (0.03)(0.03)(0.03) \\
&= 2.7x10^{-5}
\end{aligned}
$$

The probability that all three clocks will fail is approximately 0.000027 or 0.0027%. It is very unlikely that all three alarm clocks will fail.

At Least Once Rule for Independent Events:

Many times we need to calculate the probability that an event will happen at least once in many trials. The calculation can get quite complicated if there are more than a couple of trials. Using the complement to calculate the probability can simplify the problem considerably. The following example will help you understand the formula.

Example 3.2.9: At Least Once Rule

The probability that a child forgets her homework on a given day is 0.15. What is the probability that she will forget her homework at least once in the next five days?

Assume that whether she forgets or not one day has no effect on whether she forgets or not the second day.

If P(forgets) = 0.15, then P(not forgets) = 0.85.

$$\begin{aligned}
&P(\text{forgets at least once in 5 tries})\\
&= P(\text{forgets 1, 2, 3, 4 or 5 times in 5 tries})\\
&= 1 - P(\text{forgets 0 times in 5 tries})\\
&= 1 - P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget}) \cdot P(\text{not forget})\\
&= 1 - (0.85)(0.85)(0.85)(0.85)(0.85)\\
&= 1 - (0.85)^5 = 0.556
\end{aligned}$$

The probability that the child will forget her homework at least one day in the next five days is 0.556 or 55.6%

The idea in Example 3.2.9 can be generalized to get the At Least Once Rule.

At Least Once Rule
If an experiment is repeated n times, the n trials are independent and the probability of event A occurring one time is P(A) then the probability that A occurs at least one time is: $P(A \text{ occurs at least once in n trials}) = 1 - P(\overline{A})^n$

Example 3.2.10: At Least Once Rule for Bird Watching

The probability of seeing a falcon near the lake during a day of bird watching is 0.21. What is the probability that a birdwatcher will see a falcon at least once in eight trips to the lake?

Let A be the event that he sees a falcon so P(A) = 0.21. Then,
$P(\overline{A}) = 1 - 0.21 = 0.79$.

$$\begin{aligned}
P(\text{at least once in eight tries}) &= 1 - P(\overline{A})^8\\
&= 1 - (0.79)^8\\
&= 1 - 0.152 = 0.848
\end{aligned}$$

The probability of seeing a falcon at least once in eight trips to the lake is approximately 0.848 or 84.8%.

Example 3.2.11: At Least Once Rule for Guessing on Multiple Choice Tests

A multiple choice test consists of six questions. Each question has four choices for answers, only one of which is correct. A student guesses on all six questions. What is the probability that he gets at least one answer correct?

Let A be the event that the answer to a question is correct. Since each question has four choices and only one correct choice, $P(\text{correct}) = \frac{1}{4}$.

That means $P(\text{not correct}) = 1 - \frac{1}{4} = \frac{3}{4}$.

$$\begin{aligned} P(\text{at least one correct in six trials}) &= 1 - P(\text{not correct})^6 \\ &= 1 - \left(\frac{3}{4}\right)^6 \\ &= 1 - 0.178 = 0.822 \end{aligned}$$

The probability that he gets at least one answer correct is 0.822 or 82.2%.

"And" Probabilities from Two-Way Tables:

"And" probabilities are usually done by one of two methods. If you know the events are independent you can use the rule $P(A \text{ and } B) = P(A) \cdot P(B)$. If the events are not independent you can use the conditional probabilities in Section 3.3. There is an exception when we have data given in a two-way table. We can calculate "and" probabilities without knowing if the events are independent or not.

Example 3.2.12: "And" Probability from a Two-Way Table

Continuation of Example 3.2.5:

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

A person is chosen at random. Find the probability that the person:

a. bought a new car and was satisfied.

$$P(\text{new car and satisfied}) = \frac{\text{number of new car and satisfied}}{\text{number of people}} = \frac{92}{250} = 0.368 = 36.8\%$$

b. bought a used car and was not satisfied.

$$P(\text{used car and not satisfied}) = \frac{\text{number of used and not satisfied}}{\text{number of people}} = \frac{47}{250} = 0.188 = 18.8\%$$

Section 3.3: Conditional Probabilities

What do you think the probability is that a man is over six feet tall? If you knew that both his parents were tall would you change your estimate of the probability? A conditional probability is a probability that is based on some prior knowledge.

A **conditional probability** is the probability that an event will occur if some other condition has already occurred. This is denoted by $P(A \mid B)$, which is read "the probability of *A* given *B*."

Example 3.3.1: Conditional Probability for Drawing Cards without Replacement

Two cards are drawn from a well shuffled deck of 52 cards without replacement. Find the following probabilities.

a. The probability that the second card is a heart given that the first card is a spade.

Without replacement means that the first card is set aside before the second card is drawn and we assume the first card is a spade. There are only 51 cards to choose from for the second card. Thirteen of those cards are hearts.

It's important to notice that the question only asks about the second card.

$$P(\text{2nd heart} \mid \text{1st spade}) = \frac{13}{51}$$

The probability that the second card is a heart given that the first card is a spade is $\frac{13}{51}$.

b. The probability that the first card is a face card and the second card an ace.

Notice that this time the question asks about both of the cards.

There are 12 face cards out of 52 cards when we draw the first card. We set the first card aside and assume that it is a face card. Then there are four aces out of the 51 remaining cards. We want to draw a face card and an ace so use multiplication.

$$P(\text{1st face card and 2nd ace}) = \frac{12}{52} \cdot \frac{4}{51} = \frac{48}{2652} \approx 0.018$$

The probability that the first card is a face card and the second card an ace is approximately 0.018 or 1.8%.

c. The probability that one card is a heart and the other a club.

There are two ways for this to happen. We could get a heart first and a club second or we could get the club first and the heart second.

$$\begin{aligned} P(\text{heart and club}) &= P(\text{heart 1st and club 2nd or club 1st and heart 2nd}) \\ &= P(\text{heart 1st and club 2nd}) + P(\text{club 1st and heart 2nd}) \\ &= \frac{13}{52} \cdot \frac{13}{51} + \frac{13}{52} \cdot \frac{13}{51} \\ &\approx 0.127 \end{aligned}$$

The probability that one card is a heart and the other a club is approximately 0.127 or 12.7%.

Example 3.3.2: Conditional Probability for Rolling Dice

Two fair dice are rolled and the sum of the numbers is observed. What is the probability that the sum is at least nine if it is known that one of the dice shows a five?

Since we are given that one of the dice shows a five this is a conditional probability. List the pairs of dice with one die showing a five. Be careful not to count (5,5) twice.

{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5), (5,1), (5,2), (5,3), (5,4), (5,6)}

List the pairs from above that have a sum of at least nine.

{(4,5), (5,5), (6,5), (5,4), (5,6)}

There are 11 ways for one die to show a five and five of these ways have a sum of at least nine.

$$P(\text{sum at least 9} \mid \text{one die is a 5}) = \frac{5}{11}$$

The probability that the sum is at least nine if it is known that one of the dice shows a five is $\frac{5}{11}$.

Multiplication Rule for "And" Probabilities: Any Events

For events A and B, $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

Conditional Probability

For events A and B, $P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$

Example 3.3.3: Conditional Probability for Satisfaction of Car Buyers

Two hundred fifty people who recently purchased a car were questioned and the results are summarized in the following table from Section 3.2.

Table 3.2.2: Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

What is the probability that a person is satisfied if it is known that the person bought a used car?

This is a conditional probability because we already know that the person bought a used car.

$$P(\text{satisfied} \mid \text{used car}) = \frac{P(\text{satisfied and used})}{P(\text{used})}$$

$$= \frac{\frac{83}{250}}{\frac{130}{250}} = \frac{83}{\cancel{250}} \bullet \frac{\cancel{250}}{130} = \frac{83}{130} \approx 0.638$$

The probability that a person is satisfied if it is known that the person bought a used car is approximately 0.638 or 63.8%.

Example 3.3.4: Conditional Probability for Residence and Class Standing

A survey of 350 students at a university revealed the following data about class standing and place of residence.

Table 3.3.1: Housing by Class

Residence\Class	Freshman	Sophomore	Junior	Senior	Row Totals
Dormitory	89	34	46	15	184
Apartment	32	17	22	48	119
With Parents	13	31	3	0	47
Column Totals	134	82	71	63	350

What is the probability that a student is a sophomore if that student lives in an apartment?

This is a conditional probability because we are given that the student lives in an apartment.

$$P(\text{sophomore} \mid \text{apartment}) = \frac{P(\text{sophomore and apartment})}{P(\text{apartment})}$$

$$= \frac{\frac{17}{350}}{\frac{119}{350}} = \frac{17}{\cancel{350}} \bullet \frac{\cancel{350}}{119} = \frac{17}{119} \approx 0.143$$

The probability that a student is a sophomore given that the student lives in an apartment is approximately 0.143 or 14.3%.

Section 3.4: Expected Value and Law of Large Numbers

Would you buy a lottery ticket with the numbers 1, 2, 3, 4, 5? Do you think that a winning ticket with five consecutive numbers is less likely than a winning ticket with the numbers 2, 14, 18, 23 and 32? If you are playing a slot machine in Las Vegas and you have lost the last 10 times, do you keep playing the same machine because you are "due for a win?" Have you ever wondered how a casino can afford to offer meals and rooms at such cheap rates? Should you play a game of chance at a carnival? How much should an organization charge for raffle tickets for their next fund raiser? All of these questions can be answered using probabilities.

Suppose the random variable x can take on the n values $x_1, x_2, x_3, \ldots, x_n$. If the probability that each of these values occurs is $p_1, p_2, p_3, \ldots, p_n$, respectively, then the **expected value** of the random variable is $E(x) = x_1p_1 + x_2p_2 + x_3p_3 + \ldots + x_np_n$.

Example 3.4.1: Expected Value for Raffle Tickets

Valley View Elementary is trying to raise money to buy tablets for their classrooms. The PTA sells 2000 raffle tickets at $3 each. First prize is a flat-screen TV worth $500. Second prize is an android tablet worth $375. Third prize is an e-reader worth $200. Five $25 gift certificates will also be awarded. What are the expected winnings for a person who buys one ticket?

We need to write out the probability distribution before we find the expected value.

A total of eight tickets are winners and the other 1992 tickets are losers.

Table 3.4.1: Probability Distribution for the Valley View Raffle

Outcome	Win $500	Win $375	Win $200	Win $25	Win $0
Probability	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{1}{2000}$	$\frac{5}{2000}=\frac{1}{400}$	$\frac{1992}{2000}=\frac{249}{250}$

Now use the formula for the expected value.

$$E = 500\left(\frac{1}{2000}\right) + 375\left(\frac{1}{2000}\right) + 200\left(\frac{1}{2000}\right) + 25\left(\frac{1}{400}\right) + 0\left(\frac{249}{250}\right)$$
$$= 0.60$$

It costs \$3 to buy a ticket but we only win an average of \$0.60 per ticket. That means the expected winnings per ticket are \$0.60 - \$3 = -\$2.40.

We would expect to lose an average of \$2.40 for each ticket bought. This means that the school will earn an average of \$2.40 for each ticket bought for a profit of \$2.40 • 2000 = \$4800.

Example 3.4.2: Expected Value for Profit from a Purchase

A real estate investor buys a parcel of land for \$150,000. He estimates the probability that he can sell it for \$200,000 to be 0.40, the probability that he can sell it for \$160,000 to be 0.45 and the probability that he can sell it for \$125,000 to be 0.15. What is the expected profit for this purchase?

Find the profit for each situation first. \$200,000 – \$150,000 = \$50,000 profit, \$160,000 - \$150,000 = \$10,000 profit, and \$125,000 - \$150,000 = -\$25,000 profit (loss).

The probability distribution is

Table 3.4.2: Probability Distribution for a Real Estate Purchase

Outcome	\$50,000	\$10,000	-\$25,000
Probability	0.40	0.45	0.15

$$\begin{aligned} E &= 50,000(0.40) + 10,000(0.45) + (-25,000)(0.15) \\ &= \$20,750 \end{aligned}$$

The expected profit from the purchase is \$20,750.

Example 3.4.3: Expected Value for Life Insurance

The cost of a \$50,000 life insurance policy is \$150 per year for a person who is 21-years old. Assume the probability that a person will die at age 21 is 0.001. What is the company's expected profit if the company sells 10,000 policies to 21-year olds?

There are two outcomes. If the person lives the insurance company makes a profit of \$150. The probability that the person lives is 1-0.001=0.999. If the person dies the company takes in \$150 and pays out \$50,000 for a loss of \$49,850.

Table 3.4.3: Probability Distribution for Life Insurance

Outcome	\$150	-\$49,850
Probability	0.999	0.001

The expected value for one policy is:

$$E(x) = \$150(0.999) + (-\$49,850)(0.001) = \$149.80$$

If the company sells 10,000 policies at a profit of \$149.80 each, the total expected profit is $\$149.80(10,000) = \$1,498,000$.

A game that has an expected value of zero is called a **fair game.**

Example 3.4.4: Expected Value for a Carnival Game

A carnival game consists of drawing two balls without replacement from a bag containing five red and eight white balls. If both balls are red you win \$6.00. If both balls are white you lose \$1.50. Otherwise you lose \$1.00. Is this a fair game? What would you expect would happen if you played the game many times?

First we need to find the probability distribution. These are conditional probabilities since the balls are drawn without replacement.

$$P(\text{both red}) = \frac{5}{13}\cdot\frac{4}{12} = \frac{20}{156} = \frac{5}{39}, \quad P(\text{both white}) = \frac{8}{13}\cdot\frac{7}{12} = \frac{56}{156} = \frac{14}{39}$$

$$\begin{aligned} P(\text{1 red and 1 white}) &= P(\{\text{red then white}\} \text{ or } \{\text{white then red}\}) \\ &= P(\text{red then white}) + P(\text{white then red}) \\ &= \frac{5}{13}\cdot\frac{8}{12} + \frac{8}{13}\cdot\frac{5}{12} \\ &= \frac{80}{156} = \frac{20}{39} \end{aligned}$$

Check that the sum of the probabilities is 1.00. $\frac{5}{39} + \frac{14}{39} + \frac{20}{39} = \frac{39}{39} = 1.00$

Thus, the probability distribution is valid and is shown below:

Table 3.4.4: Probability Distribution for a Carnival Game

Outcome	Win $6.00	Lose $1.50	Lose $1.00
Probability	$\frac{5}{39}$	$\frac{14}{39}$	$\frac{20}{39}$

Now find the expected value: $E=\frac{5}{39}(6.00)+\frac{14}{39}(-1.50)+\frac{20}{39}(-1.00)\approx -0.28$

Since the expected value is not zero this is not a fair game. Drawing two balls out of the bag is a random experiment so we cannot predict what will happen if we play the game once. We can predict what will happen if we play the game many times. We would expect to lose an average of $0.28 for every game we play. That means that the carnival will make an average of $0.28 for every game played. In this example we would refer to the carnival as the "house."

Have you ever wondered how casinos make money when they advertise a 99% payback on their slot machines? The games in a casino are not fair games since the expected value is not zero. The expected value of the game for a gambler is a small negative number like -$0.01. For a particular game the gambler may win or the gambler may lose. It's a random experiment and we cannot predict the outcome. What we can predict is what will happen if the gambler continues to play the game many times. If the expected value is -$0.01, the gambler will expect to lose an average of $0.01 for every game played. If he/she plays 100 games, he/she will expect to lose 100(0.01) = $1.00. Every penny the gambler loses the casino keeps. If hundreds of gamblers play hundreds of games each, every day of the year all those pennies add up to millions of dollars. The casino is referred to as the "house" and the $0.01 that the house expects to win for each game played is called the "house edge."

The **house edge** is the amount that the house can expect to earn for each dollar bet.

Figure 3.4.5: Roulette Wheel

(Ogle, 2009)

Example 3.4.5: House Edge in Roulette

A roulette wheel consists of 38 slots numbered 0, 00, and 1 through 36, evenly spaced around a wheel. The wheel is spun one direction and a ball is rolled around the wheel in the opposite direction. Eventually the ball will drop into one of the numbered slots. A player bets $1 on a single number. If the ball lands in the slot for that number, the player wins $35, otherwise the player loses the $1. Find the house edge for this type of bet.

The house edge is the expected value so we need to find the probability distribution and then the expected value.

There are 38 slots. One slot wins and the other 37 slots lose so

$P(\text{win})=\frac{1}{38}$ and $P(\text{lose})=\frac{37}{38}$.

The probability distribution is

Table 3.4.6: Probability Distribution for a Roulette Wheel

Outcome	Win \$35	Lose \$1
Probability	$\frac{1}{38}$	$\frac{37}{38}$

The expected value is:

$$E=\$35\left(\frac{1}{38}\right)+(-1)\left(\frac{37}{38}\right)\approx -\$0.0526$$

The expected value of the game is -$0.0526. This means that the player would expect to lose an average of 5.26 cents for each game played. The house would win an average of 5.26 cents for each game played.

The house edge is 5.26 cents.

Gamblers' Fallacy and Streaks:

Often times a gambler on a losing streak will keep betting in the belief that his/her luck must soon change. Consider flipping a fair coin. Each toss of the coin is independent of all the other tosses. Assume the coin has landed heads up the last eight times. Some people erroneously believe that the coin is more likely to land tails up on the next toss. In reality, the coin still has a 50% chance of landing tails up. It does not matter what happened the last eight tosses.

The **gambler's fallacy** is the mistaken belief that a streak of bad luck makes a person due for a streak of good luck.

Example 3.4.6: Streaks

Toss a fair coin seven times and record which side lands up. For example HHTHTTT would represent getting a head on the first, second, and fourth tosses and a tail on the other tosses. Is a streak of all heads less likely than the other possible outcomes?

As we will see in Section 3.5, there are 128 possible ways to toss a coin seven times. Some of the possibilities are HHTTHHT, HTHTHTH, HHHHTTT, and HTTHTTH. Because tossing coins are independent events and the coin is fair each of these 128 possibilities has the same probability.

$P(\text{HHTTHHT}) = \frac{1}{128}$, $P(\text{HTHTHTH}) = \frac{1}{128}$, etc… This also means that the probability of getting all heads is $P(\text{HHHHHHH}) = \frac{1}{128}$. Getting a streak of all heads has exactly the same probability as any other possible outcome.

Law of Large Numbers:

When studying probabilities, many times the law of large numbers will apply. If you want to observe what the probability is of getting tails up when flipping a coin, you could do an experiment. Suppose you flip a coin 20 times and the coin comes up tails nine times. Then, using an empirical probability, the probability of getting tails is 9/20 = 45%. However, we know that the theoretical probability for getting tails should be 1/2 = 50%. Why is this different? It is because there is error inherent to sampling methods. However, if you flip the coin 100 times or 1000 times, and use the information to calculate an empirical probability for getting tails up, then the probabilities you will observe will become closer to the theoretical probability of 50%. This is the law of large numbers.

The **law of large numbers** means that with larger numbers of trials of an experiment the observed empirical probability of an event will approach the calculated theoretical probability of the same event.

Section 3.5: Counting Methods

Recall that $P(A)=\dfrac{\text{number of ways for } A \text{ to occur}}{\text{total number of outcomes}}$ for theoretical probabilities. So far the problems we have looked at had rather small total number of outcomes. We could easily count the number of elements in the sample space. If there are a large number of elements in the sample space we can use counting techniques such as permutations or combinations to count them.

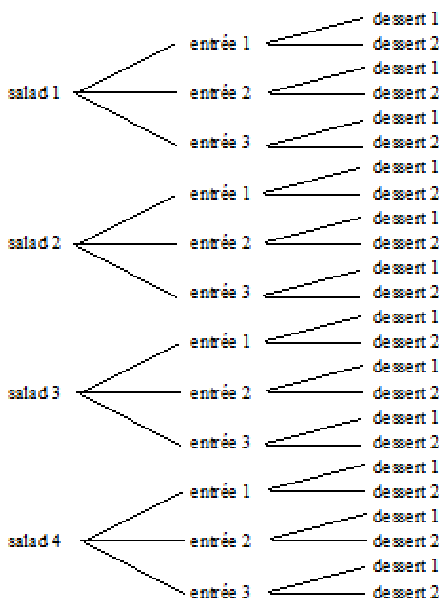
Multiplication Principle and Tree Diagrams:

The simplest of the counting techniques is the multiplication principle. A tree diagram is a useful tool for visualizing the multiplication principle.

Example 3.5.1: Multiplication Principle for a Three Course Dinner

Let's say that a person walks into a restaurant for a three course dinner. There are four different salads, three different entrees, and two different desserts to choose from. Assuming the person wants to eat a salad, an entrée and a desert, how many different meals are possible?

Figure 3.5.1: Tree Diagram for Three-Course Dinner



Looking at the tree diagram we can see that the total number of meals is 4 x 3 x 2 = 24 meals.

Multiplication Principle: If there are n_1 ways to of choosing the first item, n_2 ways of choosing the second item after the first item is chosen, n_3 ways of choosing the third item after the first two have been chosen, and so on until there are n_k ways of choosing the last item after the earlier choices, then the total number of choices overall is given by $n_1 \bullet n_2 \bullet n_3 \bullet \ldots \bullet n_k$.

Example 3.5.2: Multiplication Principle for Lining up People

Let's look at the number of ways that four people can line up. We can choose any of the four people to be first. Then there are three people who can be second and two people who can be third. At this point there is only one person left to be last. Using the multiplication principle there are 4 x 3 x 2 x 1 = 24 ways for four people to line up.

This type of calculation occurs frequently in counting problems so we have some notation to simplify the problem.

The **factorial** of n, read "n factorial" is $n! = n(n-1)(n-2)...(2)(1)$.
By definition, 0! = 1.

Example 3.5.3: Factorials

5! = 5 x 4 x 3 x 2 x 1 = 120

8! = 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 40,320

Factorials get very large very fast. 20! = 2.43 x 10^{18} and 40! = 8.16 x 10^{47}. 70! is larger than most calculators can handle.

The multiplication principle may seem like a very simple idea but it is very powerful. Many complex counting problems can be solved using the multiplication principle.

Example 3.5.4: Multiplication Principle for License Plates

Some license plates in Arizona consist of three digits followed by three letters. How many license plates of this type are possible if:

a. both digits and letters can be repeated?

There are 10 digits (0, 1, 2, 3, …, 9) and 26 letters.

$$\underbrace{(\underline{10}\cdot\underline{10}\cdot\underline{10})}_{\text{digits}}\cdot\underbrace{(\underline{26}\cdot\underline{26}\cdot\underline{26})}_{\text{letters}}=17,576,000 \text{ license plates}$$

b. letters can be repeated but digits cannot?

$$\underbrace{(\underline{10}\cdot\underline{9}\cdot\underline{8})}_{\text{digits}}\cdot\underbrace{(\underline{26}\cdot\underline{26}\cdot\underline{26})}_{\text{letters}}=12,654,720 \text{ license plates}$$

c. the first digit cannot be zero and both digits and letters can be repeated?

$$\underbrace{(\underline{9}\cdot\underline{10}\cdot\underline{10})}_{\text{digits}}\cdot\underbrace{(\underline{26}\cdot\underline{26}\cdot\underline{26})}_{\text{letters}}=15,818,400 \text{ license plates}$$

d. neither digits nor numbers can be repeated.

$$\underbrace{(\underline{10}\cdot\underline{9}\cdot\underline{8})}_{\text{digits}}\cdot\underbrace{(\underline{26}\cdot\underline{25}\cdot\underline{24})}_{\text{letters}}=11,232,000 \text{ license plates}$$

Permutations:

Consider the following counting problems. 1) In how many ways can three runners finish a race? 2) In how many ways can a group of three people be chosen to work on a project? What is the difference between these two problems? In the first problem the order that the runners finish the race matters. In the second problem the order in which the three people are chosen is not important, only which three people are chosen matters.

A **permutation** is an arrangement of a set of items.
The number of permutations of n items taking r at a time is given by:

$$P(n,r)=\frac{n!}{(n-r)!}$$

Note: Many calculators can calculate permutations directly. Look for a function that looks like ${}_nP_r$ *or* $P(n,r)$.

Example 3.5.5: Permutation for Race Cars

Let's look at a simple example to understand the formula for the number of permutations of a set of objects. Assume that 10 cars are in a race. In how many ways can three cars finish in first, second and third place? The order in which the cars finish is important. Use the multiplication principle. There are 10 possible cars to finish first. Once a car has finished first, there are nine cars to finish second. After the second car is finished, any of the eight remaining cars can finish third. 10 x 9 x 8 = 720. This is a permutation of 10 items taking three at a time.

Using the permutation formula:

$$P(10,3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \bullet 9 \bullet 8 \bullet \cancel{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{\cancel{7 \bullet 6 \bullet 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}} = 10 \bullet 9 \bullet 8 = 720$$

Using the multiplication principle:

$\underline{10} \bullet \underline{9} \bullet \underline{8} = 720$

There are 720 different ways for cars to finish in the top three places.

Example 3.5.6: Permutation for Orchestra Programs

The school orchestra is planning to play six pieces of music at their next concert. How many different programs are possible?

This is a permutation because they are arranging the songs in order to make the program.

Using the permutation formula:

$$P(6,6) = \frac{6!}{(6-6)!} = \frac{6!}{0!} = \frac{720}{1} = 720$$

Using the multiplication principle:

$\underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 720$

There are 720 different ways of arranging the songs to make the program.

Example 3.5.7: Permutation for Club Officers

The Volunteer Club has 18 members. An election is held to choose a president, vice-president and secretary. In how many ways can the three officers be chosen?

The order in which the officers are chosen matters so this is a permutation.

Using the permutation formula:

$$P(18,3) = \frac{18!}{(18-3)!} = \frac{18!}{15!} = 18 \bullet 17 \bullet 16 = 4896$$

Note: All digits in 18! in the numerator from 15 down to one will cancel with the 15! in the denominator.

Using the multiplication principle:

$$\underset{\text{Pres.}}{\underline{18}} \cdot \underset{\text{V.P.}}{\underline{17}} \cdot \underset{\text{Sec.}}{\underline{16}} = 4896$$

There are 4896 different ways the three officers can be chosen.

Another notation for permutations is *nPr*. So, $P(18,3)$ can also be written as ${}_{18}P_3$. Most scientific calculators have an *nPr* button or function.

Combinations:

Example 3.5.8: Formula for Combinations

Choose a committee of two people from persons A, B, C, D and E. By the multiplication principle there are $5 \cdot 4 = 20$ ways to arrange the two people.

AB	AC	AD	AE	BA	BC	BD	BE	CA	CB
CD	CE	DA	DB	DC	DE	EA	EB	EC	ED

Committees AB and BA are the same committee. Similarly for committees CD and DC. Every committee is counted twice. $\frac{20}{2} = 10$ so there are 10 possible different committees.

Now choose a committee of three people from persons A, B, C, D and E. There are $5 \cdot 4 \cdot 3 = 60$ ways to pick three people in order. Think about the committees with persons A, B and C. There are $3! = 6$ of them.

ABC ACB BAC BCA CAB CBA

Each of these is counted as one of the 60 possibilities but they are the same committee. Each committee is counted six times so there are $\frac{60}{6} = 10$ different committees.

In both cases we divided the number of permutations by the number of ways to rearrange the people chosen.

The number of permutations of n people taking r at a time is *P(n,r)* and the number of ways to rearrange the people chosen is *r!*. Putting these together we get

$$\frac{\text{\# permutations of n items taking r at a time}}{\text{\# ways to arrange r items}} = \frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{1}}$$

$$= \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$$

$$= \frac{n!}{(n-r)!r!}$$

A **combination** is a selection of objects in which the order of selection does not matter.

The number of combinations of n items taking r at a time is: $C(n,r) = \dfrac{n!}{r!(n-r)!}$

Note: Many calculators can calculate combinations directly. Look for a function that looks like $_nC_r$ *or* $C(n,r)$.

Example 3.5.9: Combination for Picking Books

A student has a summer reading list of eight books. The student must read five of the books before the end of the summer. In how many ways can the student read five of the eight books?

The order of the books is not important, only which books are read. This is a combination of eight items taking five at a time.

$$C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8 \bullet 7 \bullet 6 \bullet \cancel{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1}}{\cancel{5 \bullet 4 \bullet 3 \bullet 2 \bullet 1} \bullet 3 \bullet 2 \bullet 1} = \frac{8 \bullet 7 \bullet \cancel{6}}{\cancel{3 \bullet 2} \bullet 1} = 8 \bullet 7 = 56$$

There are 56 ways to choose five of the books to read.

Example 3.5.10: Combination for Halloween Candy

A child wants to pick three pieces of Halloween candy to take in her school lunch box. If there are 13 pieces of candy to choose from, how many ways can she pick the three pieces?

This is a combination because it does not matter in what order the candy is chosen.

$$C(13,3)=\frac{13!}{3!(13-3)!}=\frac{13!}{3!10!}=\frac{13\bullet 12\bullet 11\bullet \cancel{10\bullet 9\bullet 8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2\bullet 1}}{3\bullet 2\bullet 1\bullet \cancel{10\bullet 9\bullet 8\bullet 7\bullet 6\bullet 5\bullet 4\bullet 3\bullet 2\bullet 1}}$$

$$=\frac{13\bullet 12\bullet 11}{3\bullet 2\bullet 1}=\frac{1716}{6}=286$$

There are 286 ways to choose the three pieces of candy to pack in her lunch.

Note: The difference between a combination and a permutation is whether order matters or not. If the order of the items is important, use a permutation. If the order of the items is not important, use a combination.

Example 3.5.11: Permutation or Combination for Bicycle Serial Numbers

A serial number for a particular model of bicycle consists of a letter followed by four digits and ends with two letters. Neither letters nor numbers can be repeated. How many different serial numbers are possible?

This is a permutation because the order matters.

Use the multiplication principle to solve this. There are 26 letters and 10 digits possible.

$$\underline{26}\cdot\underline{10}\cdot\underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{25}\cdot\underline{24}=78{,}624{,}000$$

There are 78,624,000 different serial numbers of this form.

Example 3.5.12: Permutation or Combination for Choosing Men and Women

A class consists of 15 men and 12 women. In how many ways can two men and two women be chosen to participate in an in-class activity?

This is a combination since the order in which the people is chosen is not important.

Choose two men: $C(15,2)=\frac{15!}{2!(15-2)!}=\frac{15!}{2!13!}=105$

Choose two women: $C(12,2)=\frac{12!}{2!(12-2)!}=\frac{12!}{2!10!}=66$

We want 2 men <u>and</u> 2 women so multiply these results.

$$105(66)=6930$$

There are 6930 ways to choose two men and two women to participate.

Probabilities Involving Permutations and Combinations:

Now that we can calculate the number of permutations or combinations, we can use that information to calculate probabilities.

Example 3.5.13: Probability with a Combination for Choosing Students

There are 20 students in a class. Twelve of the students are women. The names of the students are put into a hat and five names are drawn. What is the probability that all of the chosen students are women?

This is a combination because the order of choosing the students is not important.

$$P(\text{all females}) = \frac{\text{\# ways to pick 5 women}}{\text{\# ways to pick 5 students}}$$

The number of way to choose 5 women is

$$C(12,5) = 792$$

The number of ways to choose 5 students is

$$C(20,5) = 15{,}504$$

$$P(\text{all females}) = \frac{\text{\# ways to pick 5 women}}{\text{\# ways to pick 5 students}} = \frac{792}{15{,}504} = 0.051$$

The probability that all the chosen students are women is 0.051 or 5.1%.

Example 3.5.14: Probability with a Permutation for a Duck Race

The local Boys and Girls Club holds a duck race to raise money. Community members buy a rubber duck marked with a numeral between 1 and 30, inclusive. The box of 30 ducks is emptied into a creek and allowed to float downstream to the finish line. What is the probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively?

This is a permutation since the order of finish is important.

$$P(\text{5, 18 \& 21 finish 1st, 2nd \& 3rd}) = \frac{\text{\# ways 5, 18 \& 21 finish 1st, 2nd \& 3rd}}{\text{\# ways any ducks can finish 1st, 2nd \& 3rd}}$$

There is only one way that the ducks can finish with #5 in first, #18 in second and #21 in third.

The number of ways any ducks can finish in first, second and third is

$$P(30,3) = 24,360$$

$$P(5,\ 18\ \&\ 21\text{ finish 1st, 2nd \& 3rd}) = \frac{\#\text{ ways 5, 18 \& 21 finish 1st, 2nd \& 3rd}}{\#\text{ ways any ducks can finish 1st, 2nd \& 3rd}}$$

$$= \frac{1}{24,360} \approx 4.10x10^{-5}$$

The probability that ducks numbered 5, 18 and 21 finish in first, second and third, respectively, is approximately 0.000041 or 0.0041%.

Example 3.5.15: Probability with a Permutation for Two-Card Poker Hands

A poker hand consists of two cards. What is the probability that the poker hand consists of two jacks or two fives?

It’s not possible to get two jacks and two fives at the same time so these are mutually exclusive events.

The number of ways to get two jacks is

$C(4,2) = 6$.

The number of ways to get two fives is

$C(4,2) = 6$

The number of ways to get two jacks or two fives is

6 + 6 = 12.

The total number of ways to get a 2-card poker hand is

$C(52,2) = 1326$

$$P(2\text{ jacks or 2 fives}) = \frac{\text{number of ways to get 2 jacks or 2 fives}}{\text{number of ways to choose 2 cards}} = \frac{12}{1326} \approx 0.009$$

The probability of getting two jacks or two fives is approximately 0.009 or 0.9%.

Example 3.5.16: Probability with a Combination for Rotten Apples

A basket contains 10 good apples and two bad apples. If a distracted shopper reaches into the basket and picks three apples without looking, what is the probability he gets one bad apple?

This is a combination since the order in which the apples were picked is not important. He picks three apples total. If one apple is bad the other two must be good. Find the probability of one bad apple and two good apples.

$$P\left(\text{one bad and two good apples}\right)=\frac{\text{\# ways to get one bad and two good apples}}{\text{\# ways to get three apples}}$$

The number of ways to get one bad and two good apples is

$$C\left(2,1\right)\cdot C\left(10,2\right)=2\cdot 45=90$$

The number of ways to get three apples is

$$C\left(10,3\right)=120$$

$$P\left(\text{one bad and two good apples}\right)=\frac{\text{\# ways to get one bad and two good apples}}{\text{\# ways to get three apples}}$$

$$=\frac{90}{120}=0.75$$

The probability of getting one bad apple out of three apples is 0.75 or 75%.

Chapter 3 Homework

1. A random experiment consists of drawing a single card from a well-shuffled deck and recording the suit. Write the sample space for this experiment.

2. A random experiment consists of drawing a single card from a well-shuffled deck and recording the number. Write the sample space for this experiment.

3. A random experiment consists of tossing a fair coin five times and recording the number of tails. Write the sample space for this experiment.

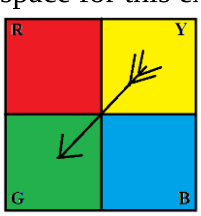
4. A random experiment consists of tossing a fair coin four times and recording the number of heads. Write the sample space for this experiment.

5. A random experiment consists of tossing three fair coins and recording whether each coin lands heads up or tails up. Write the sample space for this experiment.

6. A random experiment consists of tossing four fair coins and recording whether each coin lands heads up or tails up. Write the sample space for this experiment.

7. A random experiment consists of tossing a fair die and then tossing a fair coin. The number showing on top of the die and whether the coin lands heads up or tails up is recorded. Write the sample space for this experiment.

8. A spinner in the following figure is spun and a coin is tossed. The color of the spinner and whether the coin is heads up or tails up is recorded. Write the sample space for this experiment.



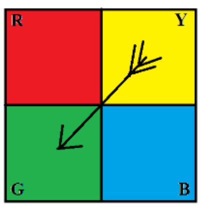
9. A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that:
    a. the card is a seven.
    b. the card is a face card.
    c. the card is a number between 2 and 5, inclusive.

10. A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that:
    a. the card is a jack.
    b. the card is a club.
    c. the card is a number less than 4, including aces.

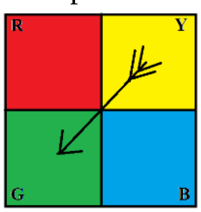
11. Two fair dice are tossed. What is the probability that the sum of the numbers is 5?

12. Two fair dice are tossed. What is the probability that the sum of the numbers is 10?

13. The spinner is spun once. What is the probability that it lands on green?



14. The spinner is spun once. What is the probability that it lands on blue?



15. An urn contains 10 red balls, 15 white balls, and 20 black balls. A single ball is selected at random. What is the probability that the ball is:
    a. red?
    b. white?
    c. not black?
    d. black or white?

16. A candy dish contains 12 chocolate candies, 18 butterscotch candies, eight caramels, and 15 peppermints. A single candy is selected at random. What is the probability that the candy is:
    a. a butterscotch candy?
    b. not a caramel candy?
    c. a chocolate or a peppermint candy?

17. Three fair coins are tossed at the same time. What is the probability of getting exactly two heads? *Hint: use problem #5.*

18. Four fair coins are tossed at the same time. What is the probability of getting exactly one head? *Hint: use problem #6.*

19. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, what is the probability that:
    a. The student is a freshman?
    b. The student is taking MAT 142?

20. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last five years. The data is listed in the following table.

Year/ Number of Bedrooms	one or two	three	four	five or more	Total
2009	5	12	25	3	45
2010	7	15	22	3	47
2011	6	18	28	6	58
2012	6	16	30	2	54
2013	5	17	28	4	54
Total	29	78	133	18	258

a. Find the probability that a randomly selected house sold by the agent had four bedrooms.
b. Find the probability that a randomly selected house sold by the agent was sold in 2010.
c. Find the probability that a randomly selected house sold by the agent had less than four bedrooms.
d. Find the probability that a randomly selected house sold by the agent was sold after 2011.

21. A student believes he has an 80% chance of passing his English class. What is the probability he will not pass the class?

22. There is a 45% chance that it will snow today. What is the probability that it will not snow today?

23. A single die is rolled. What is the probability of not rolling a five?

24. A student is randomly chosen from a class of 30 students. If seven of the students are majoring in business, what is the probability that the randomly chosen student is not majoring in business?

25. There is a 35% chance that a bus will arrive early at a bus stop. Explain why you cannot assume that there is a 65% probability that the bus will be late at the same bus stop.

26. Are each of the following valid probability distributions or not? For each one, explain why or why not.

a.

Outcome	A	B	C	D	E
Probability	0.2	0.4	-0.2	0.4	0.2

b.

Outcome	A	B	C	D	E
Probability	0.2	0.4	0.1	0.4	0.2

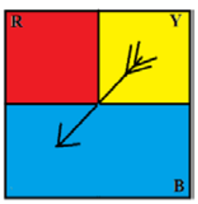
c.

Outcome	A	B	C	D	E
Probability	0.20	0.30	0.10	0.15	0.25

27. Three fair coins are tossed and the number of heads recorded. Write the probability distribution for this random experiment. *Hint: use problem #5.*

28. Four fair coins are tossed and the number of heads recorded. Write the probability distribution for this experiment. *Hint: use problem #6.*

29. The spinner in the following figure is spun once and the color recorded. Write the probability distribution for this random experiment.



30. An urn contains three red balls, four blue balls, and five green balls. A ball is selected at random and its color recorded. Write out the probability distribution for this experiment.

31. A single card is drawn from a well-shuffled deck of 52 cards. The card is either an even number, an odd number or a face card. If we consider aces to be ones, write the probability distribution for this random experiment.

32. Two fair dice are rolled. Find the odds for rolling a sum of five.

33. Two fair dice are rolled. Find the odds for rolling a sum of 10.

34. Two fair dice are rolled. Find the odds against rolling a sum of six.

35. Two fair dice are rolled. Find the odds against rolling a sum of eight.

36. A single card is drawn from a well-shuffled deck of 52 cards. Find the odds that the card is a nine.

37. A single card is drawn from a well-shuffled deck of 52 cards. Find the odds against the card being a face card.

38. A candy dish contains 12 chocolate candies, 18 butterscotch candies, 8 caramels, and 15 peppermints. A single candy is selected at random. What are the odds that the candy is a chocolate candy?

39. An urn contains 10 red balls, 15 white balls, and 20 black balls. A single ball is selected at random. Find the odds against drawing a white ball.

40. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last year. The data is listed in the following table.

Number of Bedrooms	1 or 2	3	4	5 or more
Number of Houses	5	12	25	3

If a house sold by the agent is selected at random, find:

a. the odds for the house having four bedrooms.
b. the odds for the house having less than three bedrooms.
c. the odds against the house having five or more bedrooms.
d. the odds against the house having more than three bedrooms.

41. Suppose the odds that Josh will win a tennis match against Jesse are 2 to 1. What is the probability that Josh will win?

42. The odds of winning a particular carnival game are 27 to 5. Find the probability of winning the game.

43. The odds of winning a particular carnival game are 15 to 28. Find the probability of losing the game.

44. The odds that a randomly selected student is a male are 21 to 25. Find the probability that a randomly selected student is a male.

45. A single card is drawn from a well-shuffled deck of 52 cards. Are the events E = the card is an even number and F = the card is a heart mutually exclusive? Explain why or why not.

46. A single card is drawn from a well-shuffled deck of 52 cards. Are the events E = the card is a face card and F = the card is a seven mutually exclusive? Explain why or why not.

47. An instructor randomly selects a student from his class. Are the events E = the student is taking History and F = the student is a freshman mutually exclusive? Explain why or why not?

48. A campus security officer randomly selects a car in the parking lot. Are the event E = the car is a Toyota and F = the car is red mutually exclusive? Explain why or why not.

49. Suppose you roll a single fair die. What is the probability of getting a four or a five?

50. Suppose you roll a single fair die. What is the probability of getting a two or an even?

51. Suppose you draw one card from a standard deck of cards. What is the probability that the card is an ace or a diamond?

52. Suppose you draw one card from a standard deck of cards. What is the probability that the card is an ace or the king of diamonds?

53. A teacher asks a class of 40 students about the classes they are taking. 17 of the students are taking math, 31 are taking English and 15 are taking both math and English. What is the probability that a randomly selected student is taking either math or English?

54. A teachers looks over her class and notices a few trends. Out of the 60 students in the class, 23 have brown hair, seven have green eyes, and three have both brown hair and green eyes. What is the probability that a randomly selected student has either brown hair or green eyes?

55. A student observes 46 vehicles in the CCC parking lot. He notices that 12 of the vehicles are red and that 19 of the vehicles are 4-wheel drive. If the probability that a randomly chosen vehicle is red or has 4-wheel drive is 0.609, what is the probability that the car is red and has 4-wheel drive?

56. Are drawing a card from a deck and tossing a coin independent events? Explain why or why not.

57. A single card is drawn from a deck. Are drawing a red card and drawing a heart independent events? Explain why or why not.

58. A jar contains three red, four blue, and five white marbles. A marble is drawn and its color is recorded. The marble is put back in the jar and a second marble is drawn. Is drawing two marbles in this manner independent events or not? Explain why or why not.

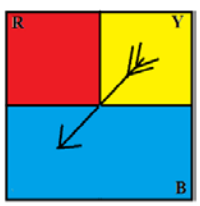
59. A jar contains three red, four blue, and five white marbles. A marble is drawn and its color is recorded. The marble is not put back in the jar before a second marble is drawn. Is drawing two marbles in this manner independent events or not? Explain why or why not.

60. A fair coin is tossed ten times. What is the probability of getting ten heads?

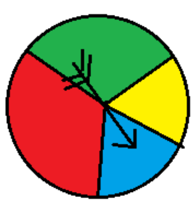
61. A fair coin is tossed then a fair die is rolled. What is the probability of getting a head and a number less than three?

62. A card is drawn from a well-shuffled deck, its number recorded, and the card returned to the deck. A second card is then drawn.
    a. What is the probability of getting a jack first and a spade second?
    b. What is the probability of getting a jack and a spade in any order?

63. The spinner shown in the following figure is spun three times. What is the probability that the spinner lands on red first, blue second, and yellow third?



64. The spinner shown in the following figure is spun four times. What is the probability that the spinner lands on red the first three times and blue the fourth time?



65. A mother figures that her son will forget his homework about 20% of the time. What is the probability that he will forget his homework at least once in the next 10 school days?

66. A bird watcher expects to see a falcon about 35% of the times he visits a nearby park. If he visits the park 12 times in the next month, what is the probability that he will see a falcon at least once?

67. A police officer watching a particular stretch of highway in Montana finds that about one out of every five drivers is speeding. What is the probability that at least one of the next eight cars he sees is speeding?

68. An instructor expects 10% of the class to earn an A on the final exam. If there are 23 students in the class, what is the probability that at least one student earns an A on the final exam?

69. A researcher surveys 150 young athletes asking about the last sport the athlete played and whether or not the athlete was injured playing that sport. The data is summarized in the following table.

Sport	Injured	Not Injured	Total
Gymnastics	16	34	50
Soccer	5	30	35
Football	27	18	45
Skiing	7	13	20
Total	55	95	150

If an athlete is selected at random, what is the probability that the athlete:

a. played soccer and was not injured?
b. did gymnastics and was injured?
c. played football or was injured?

70. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, what is the probability that:

a. the student is a freshman and taking MAT 121?
b. the student is a sophomore and taking MAT 187?
c. the student is a freshman or taking MAT 142?

71. Suppose you draw two cards from a standard deck of cards without replacement. What is the probability that:

a. both cards are Kings?
b. both cards are face cards?
c. the first card is a five and the second card is a Jack?
d. the first card is a Queen and the second card is a number less than five (count Aces as ones)?

72. Suppose you draw two cards from a standard deck of cards without replacement. What is the probability that:

a. you draw an Ace on the first card and a seven on the second card?
b. you draw a heart on the first card and a spade on the second card?
c. you draw an eight on the first card and a face card on the second card?
d. you draw two hearts in a row?

73. Suppose you pick two candies randomly from a box of candies and eat them. There are four chocolates, four caramels, and four mints. What is the probability that both candies will be chocolates?

74. A kindergarten class has 15 boys and 13 girls. The teacher calls on three students, one at a time, to line up at the board. What is the probability that the teacher calls up a boy followed by two girls?

75. A small parking lot has five black cars, seven white cars, three red cars, and four blue cars. If two cars leave in random order, what is the probability that a red car will leave first, followed by a black car?

76. A cooler contains six colas, eight root beers, and four ginger ales. Three kids grab a drink at random, one at a time.
    a. What is the probability that the first kid grabs a cola, the second kid grabs a ginger ale, and the third kid grabs a cola?
    b. What is the probability that the third kid grabs a root beer given that the first two grabbed colas?

77. An instructor collected the following data from the students in her classes.

Class / Year	Freshman	Sophomore	Total
MAT 121	43	15	58
MAT 142	35	28	63
MAT 187	27	32	59
Total	105	75	180

If a student is selected at random, find:
    a. the probability that the student is a freshman given the student is taking MAT 121.
    b. the probability that the student is taking MAT 142 given that the student is a sophomore.
    c. the probability that the student is a freshman and taking MAT 187.

78. A real estate agent has kept records of the number of bedrooms in the houses he has sold for the last five years. The data is listed in the table.

Year/ Number of Bedrooms	one or two	three	four	five or more	Total
2009	5	12	25	3	45
2010	7	15	22	3	47
2011	6	18	28	6	58
2012	6	16	30	2	54
2013	5	17	28	4	54
Total	29	78	133	18	258

If a house sold by the agent is randomly selected, find:

a. the probability that the house has three bedrooms given it was sold in 2012.
b. the probability that the house has five or more bedrooms and was sold in 2009.
c. the probability that the house has four bedrooms and was sold in 2013.
d. the probability that the house was sold in 2011 given that is has four bedrooms.

79. A single card is drawn from a well-shuffled deck of 52 cards. If the card is an ace, you win $12; otherwise, you lose $1.50.
    a. What is the expected value of this game?
    b. Explain what the expected value means in terms of the game.
    c. Is this a fair game or not?

80. Four thousand tickets are sold at $1 each for a charity raffle. Tickets are drawn at random without replacement. There is one $800 first prize, two $300 second prizes and eight $50 third prizes.
    a. What is the expected value of this raffle if you buy one ticket?
    b. Explain what the expected value means in terms of the raffle.
    c. Is this a fair game or not?

81. You and a friend are playing some of the games at the county fair. A particular game consists of drawing a single card from a standard deck. You win $5 if you draw an ace, $2 if you draw a face card, and $0.25 if any other card is drawn. It costs $1 to draw a card. Your friend thinks it is a great idea to play this game.
    a. Calculate the expected value for this game.
    b. Should your friend play the game or not?
    c. Using simple English explain to your friend why he should or should not play the game.

82. A casino game has an expected value of -$0.15. Explain what this means in a complete sentence. Do not use the words “expected value” in your explanation.

83. Two coins are flipped. You win $3 if either two heads or two tails turns up; you lose $4 if a head and a tail turn up. What is the expected value of the game?

84. Two dice are tossed at the same time. If the sum of the numbers is less than eight you win $5.50; if the sum is exactly eight you win $8; and if the sum is greater than eight you win $3. It costs $5.00 to play the game. Use the expected value to determine if you should play this game or not. Explain why or why not.

85. A lottery game consists of picking five numbers between 1 and 36, inclusive. You want to buy a ticket with the numbers 1, 2, 3, 4, and 5. Your friend laughs at you and says that those five numbers are really unlikely. What should you say to your friend to explain why he is wrong?

86. You and your friend are playing some slot machines in a casino. Your friend has lost the last 15 times he played. He asks to borrow some money because he feels his luck is about to change, that he is due to win after all those loses. What should you say to your friend to explain why he is wrong?

87. Suppose you want to create a computer password that has to be 10 digits long with each digit being chosen from the numbers 0 through 9. How many different passwords are available?

88. Jack wants to buy a new set of yard furniture. If there are three choices of tables, five choices of gliders, six choices of cushions, and two choices of umbrellas, how many different sets of yard furniture are possible?

89. The chess club is having a Sundae Night to raise money. There are three flavors of ice cream and six different toppings. A sundae consists of two scoops of ice cream and one topping. The ice cream scoops do not have to be the same flavor. How many different sundaes are possible?

90. The new iPhone comes in five different colors. There are three different sizes of memory available. The store has a selection of ten different cases to fit it. Assuming a person wants to buy an iPhone and a case, how many different phone/case combinations are possible?

91. How many ways can you line up 5 people?

92. Ten runners participate in the 100 meter hurdles. In how many ways can the medals for the top three finishers be awarded?

93. The Science club has 25 members. In how many ways can a President, Vice President, Secretary, and Treasurer be chosen?

94. Five friends are taking a road trip in a car that seats five. The car has a standard transmission so only two of the friends can drive it. How many seating arrangements are possible?

95. A night club owner is trying to arrange nine acts for a show. There are five musical numbers and four comedians. In how many ways can the show be arranged if:
    a. the acts can be in any order?
    b. the acts must alternate between musical numbers and comedians, starting with a musical number?
    c. the show must open and close with musical numbers but the remaining acts may be in any order?
96. Stanley wants to rearrange six science books and four history books on his shelf. How many arrangements are possible if:
    a. the books can be in any order?
    b. the science books are on the left and the history books are on the right?
    c. the four history books are in the middle with three science books on each end?

97. In how many ways can a five-card poker hand be drawn from a standard deck of cards?

98. In how many way can a two-card poker hand be drawn from a standard deck of cards?

99. The science club consists of 18 men and 12 women. Five members are chosen to staff the club's booth at the Science in the Park Festival. In how many ways can the five members be chosen if:
    a. any of the members can be chosen?
    b. all five members are women?
    c. exactly three men are chosen?

100. There are 28 graduate students in the department of Math and Statistics. Eighteen of the students are majoring in math and the other 10 are majoring in statistics. The department wants to send four of the graduate students to a conference. How many ways can the four students be chosen if:
    a. any of the students can be chosen?
    b. two students from each major must be chosen?
    c. at least two must be majoring in math.

101. A barrel contains 20 good peaches and four rotten peaches. A person selects three peaches at random. In how many ways can the person select:
    a. three rotten peaches?
    b. three good peaches?
    c. two good and one rotten peach?

102. A toddler is playing with some magnetic letters on the refrigerator. He has the letters l, t, a, b, and e. If he arranges the letters in a line to make a word, what is the probability that he makes the word "table"?

103. The theatre club has 14 female and nine male members. Two members are selected at random to do an acting exercise. What is the probability that both members are males?

104. Three friends, Al, Ted, and Bert run a foot race with five other boys. What is the probability that:
    a. Ted finishes first, Bert finishes second and Al finishes third?
    b. the three friends all finish in the top three places?

105. A night club owner is trying to arrange nine acts for a show. There are five musical numbers and four comedians. If the owner randomly arranges the acts, what is the probability that:
    a. the acts alternate between musical numbers and comedians, starting with a musical number?
    b. the show opens and closes with musical numbers?

106. The science club consists of 18 men and 12 women. Five members are chosen to staff the club's booth at the Science in the Park Festival. What is the probability that:
    a. all five members are women?
    b. exactly three men are chosen?

107. There are 28 graduate students in the department of Math and Statistics. Eighteen of the students are majoring in math and the other ten are majoring in statistics. The department wants to send four of the graduate students to a conference. What is the probability that:
    a. two students from each major are chosen to attend the conference?
    b. at least two who are majoring in math are chosen to attend the conference?

108. A barrel contains 20 good peaches and four bad peaches. A person selects three peaches at random. what is the probability of getting:
    a. three bad peaches?
    b. three good peaches?
    c. two good and one bad peach?